Appendix A: Coordinate systems

Electrical Engineering I · Module 92315 · Appendix A

From Fundamentals to AI Apocalypse · personal study guide

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A reference for Cartesian, cylindrical and spherical coordinates: how each describes a point, its unit vectors, how to convert between them, and the path, area and volume elements that every field integral in the course is built from. Keep it open next to the exercises.

A.1 Three ways to locate a point

Each system gives a point three numbers. **Cartesian** coordinates measure along three fixed directions. **Cylindrical** coordinates give the distance ρ from the z-axis, the angle φ around it (measured from the x-axis) and the height z. **Spherical** coordinates give the distance r from the origin, the angle ϑ down from the z-axis and the same angle φ.

System	Coordinates and ranges	Unit vectors
Cartesian	x, y, z, each from $-\infty$ to ∞	e_x, e_y, e_z
Cylindrical	$0 \le \rho < \infty$, $0 \le \varphi < 2\pi$, z from $-\infty$ to ∞	e_ρ, e_φ, e_z
Spherical	$0 \le r < \infty$, $0 \le \vartheta \le \pi$, $0 \le \varphi < 2\pi$	e_r, e_ϑ, e_φ

Cylindrical to Cartesian

$$x = \rho \cos \varphi$$
$$y = \rho \sin \varphi$$
$$z = z$$

Read it as: "x is rho cos phi, y is rho sin phi, and z stays z".
Going back: $\rho = \sqrt{x^2 + y^2}$, and φ from $\tan \varphi = y/x$, choosing the quadrant from the signs of x and y.

Spherical to Cartesian

$$x = r \sin \vartheta \cos \varphi$$
$$y = r \sin \vartheta \sin \varphi$$
$$z = r \cos \vartheta$$

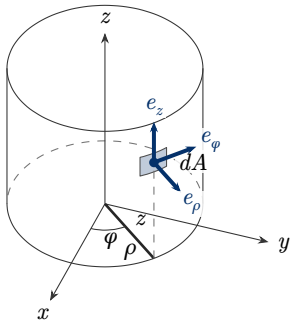
Read it as: "x is r sin theta cos phi, y is r sin theta sin phi, z is r cos theta".
$r \sin \vartheta$ is the point's distance from the z-axis, so it plays the part of ρ. Going back: $r = \sqrt{x^2 + y^2 + z^2}$, $\cos \vartheta = z/r$, and φ as for cylindrical coordinates.

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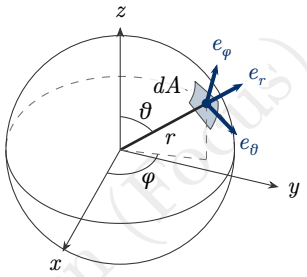
A.2 Unit vectors and cross products

In every system the three unit vectors are perpendicular to each other and have length 1, so a dot product of two different ones is 0. They form a right-handed set in the order listed: crossing one with the next gives the third, and the order runs round in a circle. Crossing in the opposite order flips the sign.

Only the Cartesian unit vectors are the same everywhere. e_ρ, e_φ, e_r and e_ϑ change direction from point to point, which matters as soon as you add or integrate vectors (A.6).



Cylindrical. P lies on the cylinder of radius ρ. Its unit vectors point away from the axis (e_ρ), around it in the direction of growing φ (e_φ) and up (e_z). The shaded patch is the wall element $\mathrm{d}A$.



Spherical. P lies on the sphere of radius r. e_r points away from the origin, e_ϑ down the line of longitude (growing ϑ) and e_φ along the line of latitude. The shaded patch is the sphere element $\mathrm{d}A$.

Cartesian

$$e_\mathrm{x} \times e_\mathrm{y} = e_\mathrm{z}$$
$$e_\mathrm{y} \times e_\mathrm{z} = e_\mathrm{x}$$
$$e_\mathrm{z} \times e_\mathrm{x} = e_\mathrm{y}$$

Read it as: "e x cross e y is e z", and so on round the circle x, y, z.
Reversed order: $e_\mathrm{y} \times e_\mathrm{x} = -e_\mathrm{z}$.

Cylindrical

$$e_\rho \times e_\varphi = e_\mathrm{z}$$
$$e_\varphi \times e_\mathrm{z} = e_\rho$$
$$e_\mathrm{z} \times e_\rho = e_\varphi$$

The circle runs ρ, φ, z. At $\varphi = 0$ the set is just e_x, e_y, e_z.

Spherical

$$e_\mathrm{r} \times e_\vartheta = e_\varphi$$
$$e_\vartheta \times e_\varphi = e_\mathrm{r}$$
$$e_\varphi \times e_\mathrm{r} = e_\vartheta$$

The circle runs r, ϑ, φ. e_φ is the same vector in cylindrical and spherical coordinates.

A.3 Converting unit vectors

Each line below writes one unit vector in another system's unit vectors. The coefficients are the cosines of the angles between the two directions, so the same numbers appear in both directions of a conversion. Use these to turn a field given in one system into another, or to add vectors that sit at different points.

Cylindrical unit vectors in Cartesian ones

$$\begin{aligned} e_\rho &= e_\mathrm{x} \cos\varphi + e_\mathrm{y} \sin\varphi \\ e_\varphi &= -e_\mathrm{x} \sin\varphi + e_\mathrm{y} \cos\varphi \\ e_\mathrm{z} &= e_\mathrm{z} \end{aligned}$$

Check at $\varphi = 90°$: $e_\rho = e_\mathrm{y}$ and $e_\varphi = -e_\mathrm{x}$, a quarter turn on from e_x, e_y.

Cartesian unit vectors in cylindrical ones

$$\begin{aligned} e_\mathrm{x} &= e_\rho \cos\varphi - e_\varphi \sin\varphi \\ e_\mathrm{y} &= e_\rho \sin\varphi + e_\varphi \cos\varphi \end{aligned}$$

The same table read the other way: the coefficient of e_φ in e_x equals the coefficient of e_x in e_φ.

Spherical unit vectors in Cartesian ones

$$\begin{aligned} e_\mathrm{r} &= e_\mathrm{x} \sin\vartheta \cos\varphi + e_\mathrm{y} \sin\vartheta \sin\varphi + e_\mathrm{z} \cos\vartheta \\ e_\vartheta &= e_\mathrm{x} \cos\vartheta \cos\varphi + e_\mathrm{y} \cos\vartheta \sin\varphi - e_\mathrm{z} \sin\vartheta \\ e_\varphi &= -e_\mathrm{x} \sin\varphi + e_\mathrm{y} \cos\varphi \end{aligned}$$

e_r is the position of P divided by r. At the north pole ($\vartheta = 0$) $e_\mathrm{r} = e_\mathrm{z}$; on the equator ($\vartheta = 90°$) $e_\vartheta = -e_\mathrm{z}$.

Cartesian unit vectors in spherical ones

$$\begin{aligned} e_\mathrm{x} &= e_\mathrm{r} \sin\vartheta \cos\varphi + e_\vartheta \cos\vartheta \cos\varphi - e_\varphi \sin\varphi \\ e_\mathrm{y} &= e_\mathrm{r} \sin\vartheta \sin\varphi + e_\vartheta \cos\vartheta \sin\varphi + e_\varphi \cos\varphi \\ e_\mathrm{z} &= e_\mathrm{r} \cos\vartheta - e_\vartheta \sin\vartheta \end{aligned}$$

Each Cartesian unit vector takes its coefficients from one column of the card above.

Between cylindrical and spherical

$$\begin{aligned} e_\rho &= e_\mathrm{r} \sin\vartheta + e_\vartheta \cos\vartheta \\ e_\mathrm{z} &= e_\mathrm{r} \cos\vartheta - e_\vartheta \sin\vartheta \end{aligned}$$

Both systems share φ and e_φ, so only a rotation by ϑ in the plane through the z-axis is left.

A.4 Position vector and its length

The position vector r points from the origin to the point. In cylindrical and spherical coordinates the angles do not appear as terms of their own: they are hidden in the directions of e_ρ and e_r, which depend on where the point is.

Position vector

$$r = x\,e_x + y\,e_y + z\,e_z$$
$$r = \rho\,e_\rho + z\,e_z$$
$$r = r\,e_r$$

Never write $\varphi\ e_\varphi$ into a position vector: an angle is not a length.

Distance from the origin

$$|r| = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2} = r$$

The distance between two points is the length of the difference of their position vectors. Form that difference in Cartesian components, because e_ρ at one point is not e_ρ at the other.

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A.5 Path, area and volume elements

Change one coordinate a little and the point moves a short distance. In Cartesian coordinates that distance is just $\mathrm{d}x$, $\mathrm{d}y$ or $\mathrm{d}z$. An angle step must be turned into a length: moving by $\mathrm{d}\varphi$ on a circle of radius ρ covers $\rho\,\mathrm{d}\varphi$, and moving by $\mathrm{d}\varphi$ on a sphere covers $r\sin\vartheta\,\mathrm{d}\varphi$, because the circle of latitude has radius $r\sin\vartheta$. Every element below is built from these three step lengths.

Connected topics: line integrals for the potential (§1.8) use $\mathrm{d}r$, charge densities (§1.6) use $\mathrm{d}V$, and flux and Gauss's law (§1.10) use $\mathrm{d}A$.

Path element

$$\begin{aligned}\mathrm{d}r &= e_\mathrm{x}\,\mathrm{d}x + e_\mathrm{y}\,\mathrm{d}y + e_\mathrm{z}\,\mathrm{d}z\\ \mathrm{d}r &= e_\rho\,\mathrm{d}\rho + e_\varphi\,\rho\,\mathrm{d}\varphi + e_\mathrm{z}\,\mathrm{d}z\\ \mathrm{d}r &= e_\mathrm{r}\,\mathrm{d}r + e_\vartheta\,r\,\mathrm{d}\vartheta + e_\varphi\,r\sin\vartheta\,\mathrm{d}\varphi\end{aligned}$$

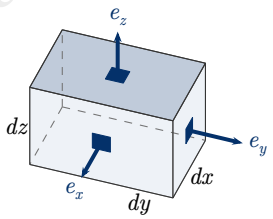
Read it as: "dr is the sum of the three step lengths, each along its unit vector".
On a path where only one coordinate changes, keep only that term: on a circle round the z-axis, $\mathrm{d}r = e_\varphi\,\rho\,\mathrm{d}\varphi$.

Volume element

$$\begin{aligned}\mathrm{d}V &= \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ \mathrm{d}V &= \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z\\ \mathrm{d}V &= r^2\sin\vartheta\,\mathrm{d}r\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\end{aligned}$$

The product of the three step lengths. Check: the ball of radius R has $V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2\sin\vartheta\;\mathrm{d}r\;\mathrm{d}\vartheta\;\mathrm{d}\varphi = 4\pi R^3/3$.

An area element is a vector: its size is the product of the two step lengths that lie in the surface, and its direction is the unit vector of the coordinate that stays fixed on the surface (outward for a closed surface).



The Cartesian volume element. Each face is a surface on which one coordinate is fixed. Its patch $\mathrm{d}A$ is the product of the two edges in the face, and its normal is the unit vector of the fixed coordinate.

Area elements, Cartesian

$$\begin{aligned}x\text{ fixed}:\ \mathrm{d}A &= e_\mathrm{x}\,\mathrm{d}y\,\mathrm{d}z\\ y\text{ fixed}:\ \mathrm{d}A &= e_\mathrm{y}\,\mathrm{d}x\,\mathrm{d}z\\ z\text{ fixed}:\ \mathrm{d}A &= e_\mathrm{z}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

On the opposite face of a box the normal points the other way, for example $-e_\mathrm{x}\;\mathrm{d}y\;\mathrm{d}z$.

Made by Thanatos A

Area elements, cylindrical

$$\text{wall, } \rho \text{ fixed}: \ \mathrm{d}A = e_\rho \, \rho \, \mathrm{d}\varphi \, \mathrm{d}z$$
$$\text{top or base, } z \text{ fixed}: \ \mathrm{d}A = \pm e_z \, \rho \, \mathrm{d}\rho \, \mathrm{d}\varphi$$
$$\text{half} - \text{plane, } \varphi \text{ fixed}: \ \mathrm{d}A = e_\varphi \, \mathrm{d}\rho \, \mathrm{d}z$$

Check: the wall of a cylinder of radius a and height h has area $2\pi a\, h$; each end disk has πa^2.

Area elements, spherical

$$\text{sphere, } r \text{ fixed}: \ \mathrm{d}A = e_r \, r^2 \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi$$
$$\text{cone, } \vartheta \text{ fixed}: \ \mathrm{d}A = e_\vartheta \, r \sin\vartheta \, \mathrm{d}r \, \mathrm{d}\varphi$$
$$\text{half} - \text{plane, } \varphi \text{ fixed}: \ \mathrm{d}A = e_\varphi \, r \, \mathrm{d}r \, \mathrm{d}\vartheta$$

Check: $\int_0^{2\pi} \int_0^{\pi} R^2 \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi = 4\pi R^2$, the area of a sphere.

A.6 Choosing a system, and three common slips

Pick the system in which the boundary of the problem is a surface where one coordinate is fixed. The integration limits then become constants, and a symmetric field has only one component.

The problem has	Use	Typical example
Flat faces, plates, a box	Cartesian	Plate capacitor, a charged slab
An axis: wires, coaxial cables, cylinders	Cylindrical	Line charge, coaxial capacitor
A centre: point charges, spheres, shells	Spherical	Point charge, charged sphere

- **Forgetting the length factor.** $\mathrm{d}V$ is $\rho \, \mathrm{d}\rho \, \mathrm{d}\varphi \, \mathrm{d}z$, not $\mathrm{d}\rho \, \mathrm{d}\varphi \, \mathrm{d}z$; the sphere's $\mathrm{d}A$ carries $r^2 \sin\vartheta$. Check the units: every $\mathrm{d}V$ must come out in m^3 and every $\mathrm{d}A$ in m^2.
- **Moving a varying unit vector out of an integral.** e_ϱ depends on φ, so $\int e_\varrho \, \mathrm{d}\varphi$ over a full circle is 0, not $2\pi e_\varrho$. Write it in Cartesian unit vectors first (A.3) and integrate the components.
- **Mixing up the two angles.** ϑ runs only from 0 to π (pole to pole), while φ runs from 0 to 2π (once round). A full sphere is not covered by letting both run to 2π.

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Personal reference sheet based on the course's coordinate-system formula collection (Lehrstuhl für Elektromagnetische Felder), rearranged and explained in my own words; the original table and figures are not reproduced. Not an official FAU publication; check details against the current course material. Made by **Thanate Amornjiraporn (Focus)**.