Exercise 1: Worked solutions

Electrical Engineering I · Module 92315 · Exercise 1

From Fundamentals to AI Apocalypse · personal study guide

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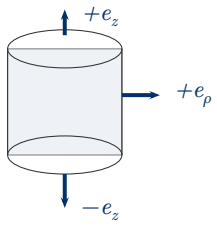
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Try every subpart before opening its answer. The steps show how the geometry determines the coordinates, how each integral is assembled, and how to check the result.

Task 1: Cylinder in cylindrical coordinates

A right circular cylinder has radius a and height h. Its axis is the z-axis, the origin is at the centre of the base, and positive z points upward.

Connected topic: Flux and Gauss's law (§1.10) use closed surfaces and their outward normals to describe which way flux crosses a boundary.



Three faces, three outward directions. The disks have normals along the axis; the curved wall points radially outward. At the rims, two faces meet.

1a: Describe the three boundary surfaces

Try first: write the coordinate ranges for the base, curved side, and top.

Answers: see "Task 1: Cylinder in cylindrical coordinates" under Answers at the end.

1b: Give the outward unit normals

Try first: at each smooth face, which way points from the cylinder's interior into the surrounding space?

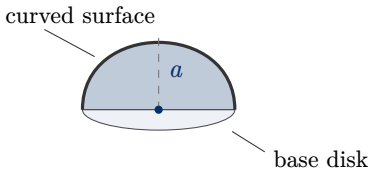
Answers: see "Task 1: Cylinder in cylindrical coordinates" under Answers at the end.

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Task 2: Hemisphere of radius *a*

Take the upper half of a sphere and include its flat circular base in the total boundary area.

Connected topic: Spherical capacitors (§1.13) use spherical geometry. The guide's sphere illustrations help distinguish curved spherical surfaces from flat circular ones.



Keep the boundary pieces separate. The curved hemisphere contributes $2\pi a^2$; the flat base disk contributes πa^2.

2a: Total surface area

Try first: find the curved area and base area separately, then add the two pieces.

Answers: see "Task 2: Hemisphere of radius a" under Answers at the end.

2b: Volume

Try first: write the spherical volume element and integrate over the upper half-sphere.

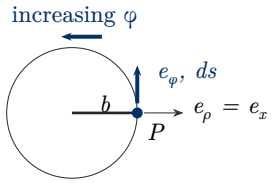
Answers: see "Task 2: Hemisphere of radius a" under Answers at the end.

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Task 3: Field integral around a circle

For each supplied field, evaluate $\oint H \cdot \mathrm{d}s$ around a circle of radius b centred at the origin. The exercise does not specify which way to traverse the loop, so assume increasing φ from 0 to 2π (counterclockwise viewed from $+z$). Reversing the direction changes the sign of any nonzero answer.

Connected topic: Electrostatic line integrals (§1.8) use the same path-vector and dot-product steps. Here H is a different supplied field; the connection is the integration method, not an identification of H with E.



At the right-hand point, the path tangent points up. A radial or x-directed field points sideways there; an azimuthal field follows the tangent. Around the full loop, the x-directed contributions cancel.

3a: Uniform x-directed field

Given $H = H_0 e_x$. Try first: parameterize the loop and express its tangent in Cartesian components.

Answers: see "Task 3: Field integral around a circle" under Answers at the end.

3b: Radial field

Given $H = H_0 e_\rho$. Try first: compare the local field direction with the tangent at every point.

Answers: see "Task 3: Field integral around a circle" under Answers at the end.

3c: Azimuthal field growing with radius

Given $H = H_0(\rho/a) e_\varphi$. Try first: evaluate this field on the circle and compare it with $\mathrm{d}s$.

Answers: see "Task 3: Field integral around a circle" under Answers at the end.

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Answers

Task 1: Cylinder in cylindrical coordinates

1a · Surface coordinates

1. In cylindrical coordinates, ρ measures distance from the axis, φ gives the angle around it, and z gives height. A point has position:

$$r = \rho\, e_\rho + z\, e_z$$

The angle has no term of its own: it sits in the direction of $e_\rho = \cos\varphi\ e_x + \sin\varphi\ e_y$, which turns as you go round the axis. So r still depends on all three coordinates.

To describe a surface, hold one coordinate fixed and let the other two run over the part of the surface we want.

2. **Base:** every point is at height $z = 0$. The disk reaches from the axis to the wall, so:

$$z = 0;\ 0 \le \rho \le a;\ 0 \le \varphi < 2\pi$$

3. **Curved side:** every point is at radius $\rho = a$. Height runs from the base to the top:

$$\rho = a;\ 0 \le z \le h;\ 0 \le \varphi < 2\pi$$

4. **Top:** every point is at height $z = h$. It is another full disk:

$$z = h;\ 0 \le \rho \le a;\ 0 \le \varphi < 2\pi$$

5. **Result:** as position vectors, with the ranges above:

$$\text{base}: r = \rho\, e_\rho$$

$$\text{side}: r = a\, e_\rho + z\, e_z$$

$$\text{top}: r = \rho\, e_\rho + h\, e_z$$

In coordinates (ρ, φ, z) these are $(\rho, \varphi, 0)$, (a, φ, z) and (ρ, φ, h).

Check: only the side fixes the radius. Each disk includes the centre and reaches all the way to a.

Task 1: Cylinder in cylindrical coordinates

1b · Outward normals

1. A surface normal is perpendicular to the surface. "Outward" chooses between the two perpendicular directions by pointing away from the solid. Cylindrical coordinate unit vectors already have length 1.
2. At the base, the interior lies above the plane $z = 0$. The outward direction is therefore downward: $-\boldsymbol{e}_{\mathbf{z}}$.
3. At the top, the interior lies below $z = h$. The outward direction is upward: $+\boldsymbol{e}_{\mathbf{z}}$.
4. On the curved side, the solid occupies $\rho < a$. Increasing ρ moves away from the axis and out of the cylinder, so the outward normal is $+\boldsymbol{e}_{\boldsymbol{\rho}}$.
5. **Result:** the outward unit normals are:

$$\text{base } -e_z;\ \text{side } +e_\rho;\ \text{top } +e_z$$

Check: at either sharp rim, the base/top normal and side normal disagree. The rim has no single smooth-face normal; normals are defined face by face.

Task 2: Hemisphere of radius a

2a · Curved surface plus base

1. Use spherical coordinates with polar angle θ measured down from $+z$. The curved hemisphere lies at $r = a$, with $0 \le \theta \le \pi/2$ and $0 \le \varphi < 2\pi$.
2. A small surface patch grows with both directions around the sphere. Its area element is:

$$\mathrm{d}A = a^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi$$

3. Integrate over the curved part:

$$A_{\text{curve}} = \int_0^{2\pi} \int_0^{\pi/2} a^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi$$

4. The angular integral gives 1 and the azimuthal integral gives 2π:

$$\int_0^{\pi/2} \sin\theta \,\mathrm{d}\theta = [-\cos\theta]_0^{\pi/2} = 1; \qquad A_{\text{curve}} = 2\pi a^2$$

5. The flat base is a disk of radius a. Add the boundary pieces:

$$A_{\text{total}} = 2\pi a^2 + \pi a^2 = 3\pi a^2$$

Check: the curved half is half of $4\pi a^2$. The area has units of length squared. The shared circular edge has zero area, so it does not affect the sum.

Task 2: Hemisphere of radius a

2b · Spherical volume integral

1. Inside the hemisphere, radius runs from the centre to the surface: $0 \leq r \leq a$. The angles cover $0 \leq \theta \leq \pi/2$ and $0 \leq \varphi < 2\pi$.
2. The spherical-coordinate volume element includes the coordinate Jacobian, which accounts for angular patches widening away from the origin:

$$\mathrm{d}V \;=\; r^2 \,\sin\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\varphi$$

3. Set up the integral. The factors depend on separate variables:

$$V \;=\; \int_0^{2\pi} \int_0^{\pi/2} \int_0^{a} r^2 \,\sin\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\varphi$$

4. Evaluate each factor:

$$\int_0^{a} r^2 \,\mathrm{d}r \;=\; \frac{a^3}{3}; \qquad \int_0^{\pi/2} \sin\theta \,\mathrm{d}\theta \;=\; 1; \qquad \int_0^{2\pi} \mathrm{d}\varphi \;=\; 2\pi$$

5. **Result:**

$$V \;=\; \frac{a^3}{3}(1)(2\pi) \;=\; \frac{2\pi a^3}{3}$$

Check: it is half the full sphere's $4\pi a^3/3$ volume, and its units are length cubed.

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Task 3: Field integral around a circle

3a · Uniform field

1. Parameterize the circle by its angle:

$$r(\varphi) = b\cos\varphi\, e_{\mathrm{x}} + b\sin\varphi\, e_{\mathrm{y}}, \qquad 0 \le \varphi \le 2\pi$$

2. Differentiate with respect to the path parameter to get the tangent line element:

$$\mathrm{d}s = (-b\sin\varphi\, e_{\mathrm{x}} + b\cos\varphi\, e_{\mathrm{y}})\,\mathrm{d}\varphi = b\, e_{\varphi}\,\mathrm{d}\varphi$$

3. Take the dot product with the constant field:

$$H \cdot \mathrm{d}s = -b\, H_0 \sin\varphi\,\mathrm{d}\varphi$$

4. Integrate over the whole circle:

$$\oint H \cdot \mathrm{d}s = -b\, H_0 \int_0^{2\pi} \sin\varphi\,\mathrm{d}\varphi = b\, H_0 [\cos\varphi]_0^{2\pi}$$

5. **Result:**

$$\oint H \cdot \mathrm{d}s = 0$$

The positive and negative tangential contributions cancel over one full turn.

Check: the zero is caused by cancellation around the loop, not by the field being perpendicular to the path at every point.

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Task 3: Field integral around a circle

3b · Radial field

1. Use the same circle parameterization. At radius b, the tangent line element is:

$$\mathrm{d}s = b\, e_\varphi \, \mathrm{d}\varphi$$

2. The radial unit vector e_ρ points out from the origin, while e_φ follows the circle. In cylindrical coordinates these perpendicular directions satisfy $e_\rho \cdot e_\varphi = 0$.

3. The dot product is zero at each point because the directions are perpendicular:

$$H \cdot \mathrm{d}s = H_0 e_\rho \cdot b\, e_\varphi \, \mathrm{d}\varphi = 0$$

4. **Result:**

$$\oint H \cdot \mathrm{d}s = 0$$

Check: unlike 3a, the integrand is zero point by point. No cancellation between different parts of the circle is needed.

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Task 3: Field integral around a circle

3c · Tangential field

1. The path is the circle $\rho = b$, with φ increasing:

$$\mathrm{d}s = b\, e_\varphi \, \mathrm{d}\varphi$$

2. Substitute $\rho = b$ into the field. Its magnitude is constant along this circle:

$$H = H_0 \frac{b}{a} e_\varphi$$

3. Both vectors point along e_φ, whose dot product with itself is 1:

$$H \cdot \mathrm{d}s = H_0 \frac{b^2}{a} \, \mathrm{d}\varphi$$

4. Integrate the constant around one revolution:

$$\oint H \cdot \mathrm{d}s = \int_0^{2\pi} H_0 \frac{b^2}{a} \, \mathrm{d}\varphi = \frac{2\pi H_0 b^2}{a}$$

5. **Result:**

$$\oint H \cdot \mathrm{d}s = \frac{2\pi H_0 b^2}{a}$$

It is positive for the assumed counterclockwise direction.

Check: H_0 has units A/m, and b^2/a has units m, so the integral has units A. Reversing the loop makes it negative.

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