Exercise 3: Worked solutions

Electrical Engineering I · Module 92315 · Exercise 3

From Fundamentals to AI Apocalypse · personal study guide

Contents

Task 7: Electric field of a uniformly charged sphere **2**

Task 8: Parallel plates with a dielectric cylinder **3**

Answers **4**

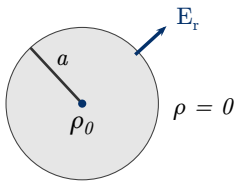
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Try each part before opening the answer. We use symmetry to choose Gaussian surfaces, then split the capacitor into the dielectric core and the air annulus.

Task 7: Electric field of a uniformly charged sphere

A spherical volume charge has density ρ_0 for $0 \leq r \leq a$ and zero outside. Its centre is the origin. Find the field everywhere in vacuum, where $D = \epsilon_0 E$.

Connected topic: Gauss's law (§1.10) turns spherical symmetry into a flux equation on a concentric sphere.



Enclosed charge depends on the Gaussian radius. Inside the material only the smaller sphere contributes; outside it, the entire charged sphere is enclosed.

7: Field inside and outside

Try first: choose a concentric spherical Gaussian surface of radius r, and find its enclosed charge separately for $r \leq a$ and $r > a$.

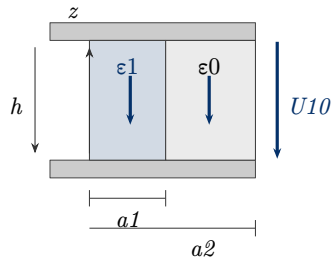
Answers: see "Task 7: Electric field of a uniformly charged sphere" under Answers at the end.

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Task 8: Parallel plates with a dielectric cylinder

Two circular metal plates have radius a_2 and separation h. A concentric dielectric cylinder of radius a_1 fills the gap between the plates; the annular region from a_1 to a_2 is air. The source assumes uniform axial fields in both regions and neglects fringing at the plate edge.

Connected topics: Capacitance (§1.13) relates charge and voltage; field energy (§1.14) gives the stored energy. Interpret ϵ_1 as the absolute permittivity of the cylinder, as distinguished from ϵ_0 for air in the figure.



Two materials share the same plate voltage. The axial field is uniform within each region; the radial material boundary separates the dielectric core from the air annulus.

The source labels the applied voltage U_{10} with an arrow from the upper plate to the lower one. Use the standard convention $U_{10} = V_1 - V_0$, upper minus lower; for positive U_{10}, the field points down, along $-e_z$.

8a: Field between the plates

Try first: relate the potential difference to the axial field, then use tangential-field continuity at the cylindrical material boundary.

Answers: see "Task 8: Parallel plates with a dielectric cylinder" under Answers at the end.

8b: Charge on the underside of the upper plate

Try first: find the free surface charge density in each patch from D = εE and the normal pointing from the upper conductor into the gap.

Answers: see "Task 8: Parallel plates with a dielectric cylinder" under Answers at the end.

8c: Capacitance

Try first: divide the upper-plate charge by the applied voltage.

Answers: see "Task 8: Parallel plates with a dielectric cylinder" under Answers at the end.

8d: Energy change when the dielectric is removed

Try first: compute the initial and final capacitances, then compare their field energies at the same fixed voltage.

Answers: see "Task 8: Parallel plates with a dielectric cylinder" under Answers at the end.

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Answers

Task 7: Electric field of a uniformly charged sphere

7 · Gauss's law in the two regions

1. Spherical symmetry makes the field radial and equally strong everywhere on a concentric Gaussian sphere. Its area is $4\pi r^2$, so the flux law reduces to:

$$\oiint D \cdot \mathrm{d}A \;=\; D_{\mathrm{r}}(r)\, 4\pi r^2 \;=\; Q_{\mathrm{enc}}(r)$$

   Since $D = \epsilon_0 E$, solving this gives E_{r} once we know the enclosed charge.

2. For $0 < r \leq a$, the Gaussian sphere contains only a smaller volume of the uniform density. The spherical volume is $4\pi r^3/3$:

$$Q_{\mathrm{enc}}(r) \;=\; \rho_0 \frac{4\pi r^3}{3}$$

3. Substitute that enclosed charge into Gauss's law and divide by $4\pi\epsilon_0 r^2$:

$$E_{\mathrm{r}}(r) \;=\; \frac{Q_{\mathrm{enc}}}{4\pi\varepsilon_0 r^2} \;=\; \frac{\rho_0 r}{3\varepsilon_0}, \qquad 0 \leq r \leq a$$

4. For $r > a$, the Gaussian sphere encloses all the charge, whose volume is the full sphere of radius a:

$$Q_{\mathrm{total}} \;=\; \rho_0 \frac{4\pi a^3}{3}; \qquad E_{\mathrm{r}}(r) \;=\; \frac{\rho_0 a^3}{3\varepsilon_0 r^2}, \qquad r > a$$

5. **Result:** for the positive density shown, the vector field points radially outward. Inside:

$$E(r) \;=\; \frac{\rho_0 r}{3\varepsilon_0} e_{\mathrm{r}}, \qquad 0 \leq r \leq a$$

   Outside, all charge is enclosed:

$$E(r) \;=\; \frac{\rho_0 a^3}{3\varepsilon_0 r^2} e_{\mathrm{r}}, \qquad r > a$$

   At the origin, $E = 0$; negative ρ_0 reverses the radial direction.

   Check: both formulas give $E(a) = \rho_0 a/(3\epsilon_0)$, so the field is continuous at the surface. Far away it falls as $1/r^2$, as if the total charge were concentrated at the centre. With SI inputs, E is in N/C.

Task 8: Parallel plates with a dielectric cylinder

8a · One voltage, two regions

1. The field is uniform and axial in each region. Moving from the lower plate at z = 0 to the upper plate at z = h gives the upper-minus-lower potential:

$$U_{10} = V_1 - V_0 = -\int_0^{\mathrm{h}} E_{\mathrm{z}}\,\mathrm{d}z$$

2. The radial boundary at $\rho = a_1$ is parallel to the z-directed field. Its tangential field component is continuous, so the assumed axial components agree:

$$E_1 = E_2 = E_{\mathrm{z}}$$

3. Because each field is uniform over the gap, the integral is $E_{\mathrm{z}}h$. Solve for the common component:

$$E_1 = E_2 = -\frac{U_{10}}{h}$$

4. **Result:** in the requested gap (not inside the metal), the field is the same everywhere:

$$E = -\frac{U_{10}}{h}\,e_z, \qquad 0 \le \rho \le a_2, \qquad 0 < z < h$$

The same field applies on both sides of $\rho = a_1$ under the uniform-field approximation. The material changes D, not this tangential E.

Check: E has units V/m. For positive upper-minus-lower voltage, potential decreases in the +z direction, so E points downward.

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Task 8: Parallel plates with a dielectric cylinder

8b · Add the two material patches

1. The outward normal from the lower face of the upper metal plate into the gap is $-e_z$. The conductor boundary relation gives free surface charge density $\sigma_f = D \cdot n$. Since $E = -(U_{10}/h)e_z$, each patch has positive σ for positive U_{10}.
2. Over the dielectric circle, $D = \epsilon_1 E$; its area is $\pi {a_1}^2$:

$$Q_{core} = \sigma_1 \pi a_1^2 = \frac{\varepsilon_1 U_{10}}{h} \pi a_1^2$$

3. Over the air annulus, $D = \epsilon_0 E$; its area is the whole plate disk minus the core disk:

$$Q_{air} = \sigma_2 \pi (a_2^2 - a_1^2) = \frac{\varepsilon_0 U_{10}}{h} \pi (a_2^2 - a_1^2)$$

4. Add the two disjoint patches on the same metal face:

$$Q = Q_{core} + Q_{air} = \frac{\pi U_{10}}{h} [\varepsilon_1 a_1^2 + \varepsilon_0 (a_2^2 - a_1^2)]$$

Result: this is the charge on the underside of the upper plate. For positive U_{10} it is positive; the lower plate carries the opposite charge.

Check: each term has permittivity × voltage × area / length, which is coulombs. If $a_1 = 0$ the full plate is air; if $a_1 = a_2$ it is fully filled by the dielectric.

Task 8: Parallel plates with a dielectric cylinder

8c · Charge-to-voltage ratio

1. Capacitance is defined by $C = Q/U$. Divide the expression from 8b by U_{10}:

$$C = \frac{\pi}{h} [\varepsilon_1 a_1^2 + \varepsilon_0 (a_2^2 - a_1^2)]$$

2. **Result:** this is the capacitance under the stated uniform-field model. The two regions behave like capacitors in parallel because they span the same voltage and their charges add.

   Check: it equals the sum of area/permittivity contributions divided by h; capacitance has units farads.

Task 8: Parallel plates with a dielectric cylinder

8d · Fixed-voltage energy change

1. Initially the capacitance is the value from 8c. After removal, the whole plate area is air:

$$C_{\mathrm{i}} = \frac{\pi}{h}[\varepsilon_1 a_1^2 + \varepsilon_0(a_2^2 - a_1^2)]; \quad C_{\mathrm{f}} = \frac{\varepsilon_0 \pi a_2^2}{h}$$

2. With voltage held fixed, stored field energy is $W = \frac{1}{2} C\ U^2$. Subtract the initial energy from the final energy:

$$\Delta W_{\text{field}} = \tfrac{1}{2}(C_{\mathrm{f}} - C_{\mathrm{i}})U_{10}^2$$

3. The air-region terms cancel, leaving the contribution from the removed core:

$$\Delta W_{\text{field}} = -\frac{\pi a_1^2(\varepsilon_1 - \varepsilon_0)}{2h} U_{10}^2$$

4. Follow the energy that leaves. With U_{10} fixed, the plate charge changes by $\Delta Q = (C_{\mathrm{f}} - C_{\mathrm{i}})U_{10} < 0$, so charge flows back into the source. The energy the source delivers is $W_{\text{source}} = U_{10}\Delta Q = 2\Delta W_{\text{field}}$, a negative number: the source takes energy back. Whoever pulls the core out supplies the difference:

$$W_{\text{mech}} = \Delta W_{\text{field}} - W_{\text{source}} = -\Delta W_{\text{field}} = \frac{\pi a_1^2(\varepsilon_1 - \varepsilon_0)}{2h} U_{10}^2$$

5. **Result:** if $\epsilon_1 > \epsilon_0$, the stored field energy decreases by this amount ($\Delta W_{\text{field}} < 0$), measured in joules. Positive mechanical work of the same size is needed to pull the core out: at fixed voltage the field draws the dielectric into the gap, and the source takes back twice the field-energy drop.

   Check: if the core is indistinguishable from air, $\varepsilon_1 = \varepsilon_0$, the change is zero. The magnitude scales with the removed area and with voltage squared.

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