Exercise 4: Worked solutions

Electrical Engineering I · Module 92315 · Exercise 4

From Fundamentals to AI Apocalypse · personal study guide

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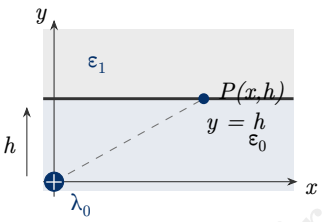
Try each part before opening the answer. We apply the interface conditions to the measured field, then use cylindrical Gauss surfaces and voltage integration for the layered coaxial geometry.

Task 9: Field at a flat dielectric boundary

An infinite line charge of density λ_0 lies along the z-axis. The lower half-space y < h is air with permittivity ϵ_0, and the upper half-space y ≥ h has permittivity ϵ_1. In the air, measurements give the field below.

$$E_0(\mathrm{x},\mathrm{y}) = \frac{\lambda_0}{\pi(\varepsilon_0 + \varepsilon_1)(\mathrm{x}^2 + \mathrm{y}^2)}\left[\mathrm{x}e_\mathrm{x} + \frac{\varepsilon_1}{\varepsilon_0}\mathrm{y}e_\mathrm{y}\right]$$

Connected topic: Material boundary conditions (§1.11) keep tangential E continuous and normal D continuous when no free surface charge is present.



Resolve the measured field into interface-parallel and interface-normal parts. The x-component stays continuous; the y-component changes with permittivity.

9: Electric field in the dielectric at y = h

Try first: evaluate the measured air field at the interface, then apply the tangential-E and normal-D boundary conditions component by component.

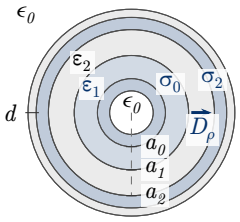
Answers: see "Task 9: Field at a flat dielectric boundary" under Answers at the end.

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Task 10: Two-dielectric coaxial capacitor

Two infinitely long concentric metal cylinders share the z-axis. The inner metal's outer radius is a_0; the outer metal's inner radius is $a_2 > a_0$. Between them, permittivity is ϵ_1 for $a_0 < \rho < a_1$ and ϵ_2 for $a_1 < \rho < a_2$. The conductor walls each have thickness d; outside the conductors is material ϵ_0. The only charges are σ_0 at the inner conductor's outer surface and σ_2 at the outer conductor's inner surface.

Connected topics: Gauss's law (§1.10) finds the radial flux; capacitance (§1.13) follows from charge per length divided by voltage.



Track each cylindrical boundary by radius. The field runs radially through ε_1 then ε_2; free surface charges sit on the two metal faces bordering the dielectric gap.

In the dielectric gap, symmetry gives $D = D_\rho(\rho)e_\rho$. The sheet fixes the voltage direction: $U_{20} = \int$ from P_2 (outer) to P_0 (inner) of $E \cdot ds$, matching the figure's arrow from the outer to the inner terminal and the course convention $U_{20} = \varphi_2 - \varphi_0$. The path therefore runs inward.

10a: Flux density, field, and outer surface charge

Try first: apply cylindrical Gauss surfaces in each dielectric, then use continuity of normal D at $\rho = a_1$ and the conductor boundary at $\rho = a_2$.

Answers: see "Task 10: Two-dielectric coaxial capacitor" under Answers at the end.

10b: Field and σ_0 as functions of U_{20}

Try first: integrate E along the path the sheet gives, inward from P_2 at a_2 to P_0 at a_0, splitting at a_1, then solve the voltage relation for σ_0.

Answers: see "Task 10: Two-dielectric coaxial capacitor" under Answers at the end.

10c: Capacitance for length Δz

Try first: use the inner conductor's charge over length Δz and divide by the voltage from 10b.

Answers: see "Task 10: Two-dielectric coaxial capacitor" under Answers at the end.

10d: Field inside the inner conductor and outside the outer one

Try first: use Gauss's law and the conductor condition $D = 0$, remembering that the two facing surface charges cancel per unit length.

Answers: see "Task 10: Two-dielectric coaxial capacitor" under Answers at the end.

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Answers

Task 9: Field at a flat dielectric boundary

9 · Apply both interface conditions

1. The interface is horizontal, so its normal is along y and its tangent is along x. There is no free charge sheet in the problem. Therefore:

$$E_{1x} = E_{0x}; \quad D_{1y} = D_{0y}, \quad \text{equivalently } \varepsilon_1 E_{1y} = \varepsilon_0 E_{0y}$$

2. First evaluate the given air field at y = h. Its tangential component is:

$$E_{0x}(x, h) = \frac{\lambda_0 x}{\pi(\varepsilon_0 + \varepsilon_1)(x^2 + h^2)}$$

3. Keep that x-component unchanged across the boundary:

$$E_{1x}(x, h) = E_{0x}(x, h)$$

4. The measured normal component contains the factor $\varepsilon_1/\varepsilon_0$. Multiplying it by $\varepsilon_0/\varepsilon_1$ to preserve D cancels that factor:

$$E_{1y}(x, h) = \frac{\varepsilon_0}{\varepsilon_1} E_{0y}(x, h) = \frac{\lambda_0 h}{\pi(\varepsilon_0 + \varepsilon_1)(x^2 + h^2)}$$

5. **Result:** combine the two components into the dielectric-side vector:

$$E_1(x, h) = \frac{\lambda_0}{\pi(\varepsilon_0 + \varepsilon_1)(x^2 + h^2)} [x e_x + h e_y]$$

Check: the tangential component is unchanged; the normal flux density matches on both sides. With SI inputs, E is in V/m (equivalently N/C).

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Task 10: Two-dielectric coaxial capacitor

10a · Use enclosed charge per unit length

1. Choose a coaxial Gaussian cylinder of radius ρ and length Δz, with $a_0 < \rho < a_2$. Radial D is constant over its curved side; its end caps contribute no flux. The enclosed free charge is the inner surface charge on length Δz:

$$D_\rho(\rho)\, 2\pi\rho\, \Delta z \;=\; \sigma_0\, 2\pi a_0\, \Delta z$$

2. Cancel the common factors. No free charge lies at the dielectric boundary $\rho = a_1$, so normal D has the same value in both layers:

$$D(\rho) \;=\; \frac{\sigma_0 a_0}{\rho}\, e_\rho, \qquad a_0 \;<\; \rho \;<\; a_2$$

3. Use D = εE separately in each material:

$$E(\rho) \;=\; \frac{\sigma_0 a_0}{\varepsilon_1 \rho} e_\rho, \qquad a_0 \;<\; \rho \;<\; a_1$$

$$E(\rho) \;=\; \frac{\sigma_0 a_0}{\varepsilon_2 \rho} e_\rho, \qquad a_1 \;<\; \rho \;<\; a_2$$

4. At the inner face of the outer metal cylinder, the normal from the conductor into the gap points inward, $-e_\rho$. Thus $\sigma_2 = D \cdot (-e_\rho)$ at $\rho = a_2$:

$$\sigma_2 \;=\; -\frac{\sigma_0 a_0}{a_2}$$

Result: D is radial and continuous through the dielectric interface; E changes magnitude inversely with permittivity. The two metal-face charges have opposite signs.

Check: $\sigma_2\ 2\pi a_2 \Delta z = -\sigma_0\ 2\pi a_0 \Delta z$, so the two facing surfaces carry equal and opposite charge.

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Task 10: Two-dielectric coaxial capacitor

10b · Integrate through both layers

1. The path runs inward, from $\rho = a_2$ to $\rho = a_0$. Along a radial path $\mathrm{d}s = e_\rho\,\mathrm{d}\rho$, and the limits carry the direction, so $E \cdot \mathrm{d}s = E_\rho\,\mathrm{d}\rho$ with ρ decreasing. Insert the two E expressions from 10a:

$$U_{20} = \int_{a_2}^{a_1} \frac{\sigma_0 a_0}{\varepsilon_2 \rho}\,\mathrm{d}\rho + \int_{a_1}^{a_0} \frac{\sigma_0 a_0}{\varepsilon_1 \rho}\,\mathrm{d}\rho$$

2. Each integral of $1/\rho$ gives a logarithm; running inward makes each one negative, for example $\int$ from a_1 to a_0 of $\mathrm{d}\rho/\rho = \ln(a_0/a_1) = -\ln(a_1/a_0)$:

$$U_{20} = -\sigma_0 a_0 \left[\frac{1}{\varepsilon_1}\ln\frac{a_1}{a_0} + \frac{1}{\varepsilon_2}\ln\frac{a_2}{a_1}\right]$$

3. Define the positive geometry/material factor:

$$S = \frac{1}{\varepsilon_1}\ln\frac{a_1}{a_0} + \frac{1}{\varepsilon_2}\ln\frac{a_2}{a_1}$$

4. Rearrange for the inner surface charge density, then substitute it back into D and E:

$$\sigma_0 = -\frac{U_{20}}{a_0 S}; \qquad D(\rho) = -\frac{U_{20}}{S\rho}e_\rho$$

$$E(\rho) = -\frac{U_{20}}{\varepsilon_1 S\rho}e_\rho, \qquad a_0 < \rho < a_1$$

$$E(\rho) = -\frac{U_{20}}{\varepsilon_2 S\rho}e_\rho, \qquad a_1 < \rho < a_2$$

5. **Result:** for positive U_{20} the outer cylinder is at the higher potential, so $\sigma_0 < 0$, $\sigma_2 = -\sigma_0 a_0/a_2 > 0$, and the field points inward, from the outer cylinder to the inner one.

   Check: the field points from high to low potential, and the outer cylinder (node 2) is high when $U_{20} > 0$, so an inward field is right. Integrating the resulting E inward from a_2 to a_0 returns $+U_{20}$. Likely mistake: integrating outward from P_0 to P_2, which flips every sign. The field has units V/m and σ_0 has units C/m².

Task 10: Two-dielectric coaxial capacitor

10c · Capacitance per unit length

1. Capacitance is the charge on the electrode at the higher potential divided by the voltage. For $U_{20} > 0$ that is the outer cylinder, whose facing surface carries $Q = \sigma_2\ 2\pi a_2 \Delta z = -\sigma_0\ 2\pi a_0 \Delta z$:

$$C = \frac{Q}{U_{20}} = \frac{-\sigma_0 2\pi a_0 \Delta z}{U_{20}}$$

2. Substitute $\sigma_0 = -U_{20}/(a_0 S)$. The signs, the voltage and a_0 cancel:

$$C = \frac{2\pi \Delta z}{S}; \quad C_B = \frac{C}{\Delta z} = \frac{2\pi}{S}$$

3. **Result:** C is the capacitance over length Δz; C_B is the capacitance loading per unit length.

   Check: S has units m/F, so C is in farads and C_B in F/m. If $\varepsilon_1 = \varepsilon_2 = \varepsilon$, then $S = \ln(a_2/a_0)/\varepsilon$, recovering the homogeneous coaxial result.

Task 10: Two-dielectric coaxial capacitor

10d · Zero field outside the charged gap

1. For $0 \le \rho < a_0$, a coaxial Gaussian cylinder encloses no free charge: σ_0 lies at $\rho = a_0$, outside this Gaussian surface. Thus its radial flux and field are zero. Within the inner metal wall D is also zero by the conductor condition.

2. For $\rho > a_2$, a Gaussian cylinder outside both facing surfaces encloses their total charge per length:

$$\frac{q_{enc}}{\Delta z} = 2\pi a_0 \sigma_0 + 2\pi a_2 \sigma_2 = 0$$

   The cancellation uses $\sigma_2 = -\sigma_0 a_0/a_2$. Therefore D and E vanish outside the outer conductor; they also vanish within its metal wall.

3. **Result:**

$$E = 0, \quad 0 \le \rho < a_0$$

$$E = 0, \quad \rho > a_2$$

   Check: the applied voltage and field are confined to the annular dielectric gap in this ideal infinitely long, concentric arrangement.

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