Tutorial 4: Worked solutions

Electrical Engineering I · Module 92315 · Tutorial 4

From Fundamentals to AI Apocalypse · personal study guide

Contents

Task 8: Electrostatics **3**

Answers **5**

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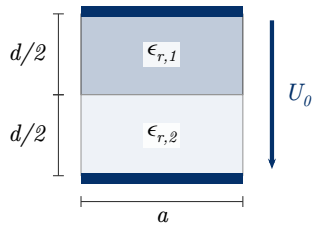
Model the initial stack as two capacitors in series. After the lower dielectric shifts, its overlap and the air-filled part form parallel branches beneath the upper dielectric.

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Task 8: Electrostatics

Two square plates of side a are separated by total distance d. Initially, dielectric 1 ($\epsilon_{r,1}$) fills the upper half and dielectric 2 ($\epsilon_{r,2}$) fills the lower half, each of thickness $d/2$. Edge effects are neglected; the exposed surrounding region is air with relative permittivity 1. The voltage arrow points downward; take positive U_0 to produce a downward field. Both relative permittivities are much greater than 1.

Connected topic: Capacitance (§1.13) relates plate geometry and permittivity to charge and voltage; the interface conditions explain how the displacement crosses the two layers.



Before the shift: two layers in series. Dielectric $\varepsilon_{r,1}$ fills the upper half and $\varepsilon_{r,2}$ the lower half, each of thickness $d/2$ over the full plate area a^2. The voltage U_0 points from the top plate to the bottom one.

8a: Find *D* and *E*

Try first: use continuity of the normal displacement at the dielectric boundary and add the two half-gap voltage drops.

Answers: see "Task 8: Electrostatics" under Answers at the end.

8b: Equivalent circuit before the shift

Try first: decide whether the layers lie one after another along the field or side by side across the plate area.

Answers: see "Task 8: Electrostatics" under Answers at the end.

8c: Total capacitance before the shift

Try first: calculate each layer's plate capacitance using area a^2 and thickness $d/2$, then combine in series.

Answers: see "Task 8: Electrostatics" under Answers at the end.

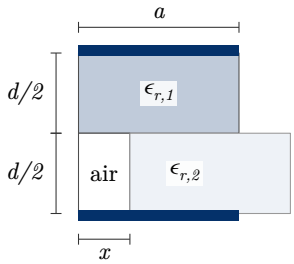
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8d: Surface charge on the lower face of the top plate

Try first: dot D with the outward normal from the metal into the dielectric at that face.

Answers: see "Task 8: Electrostatics" under Answers at the end.

The lower dielectric is now shifted out along $+e_x$ by a distance x, with $0 \le x \le a$. Its overlap width is $a-x$; the exposed gap beneath the upper dielectric is filled with air.



After the shift by x. The upper layer still covers the whole plate. Below it, a strip of width x is air and the rest, width $a - x$, is still $\varepsilon_{r,2}$; the part pushed past the plate edge carries no field.

8e: Equivalent circuit after the shift

Try first: identify the full-area layer crossed by every field line, then split the lower half into the two side-by-side regions.

Answers: see "Task 8: Electrostatics" under Answers at the end.

8f: Capacitance as a function of x

Try first: find each lower-region capacitance from its area and thickness, add the parallel branches, then combine that result in series with C_1.

Answers: see "Task 8: Electrostatics" under Answers at the end.

8g: Minimum and maximum capacitance

Try first: check how the lower parallel capacitance changes with x, then use that the series equivalent increases when either component capacitance increases.

Answers: see "Task 8: Electrostatics" under Answers at the end.

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Answers

Task 8: Electrostatics

8a · Fields in the two layers

1. With no free charge at the internal interface, the normal component of D is the same in both layers. The voltage arrow sets the field downward for positive U_0; denote the downward displacement magnitude by D_0:

$$D_1 = D_2 = -D_0\, e_z$$

2. In each linear dielectric, $D = \varepsilon_0 \varepsilon_r E$. The downward potential drop across each thickness $d/2$ adds to U_0:

$$U_0 = \frac{D_0}{\varepsilon_0\, \varepsilon_{r,1}} \cdot \frac{d}{2} + \frac{D_0}{\varepsilon_0\, \varepsilon_{r,2}} \cdot \frac{d}{2}$$

$$D_0 = \frac{2\varepsilon_0\, \varepsilon_{r,1}\, \varepsilon_{r,2}\, U_0}{d(\varepsilon_{r,1} + \varepsilon_{r,2})}$$

3. Apply the constitutive relation separately to each layer:

$$D = -\frac{2\varepsilon_0\, \varepsilon_{r,1}\, \varepsilon_{r,2}\, U_0}{d(\varepsilon_{r,1} + \varepsilon_{r,2})}\, e_z$$

$$E_1 = -\frac{2\varepsilon_{r,2}\, U_0}{d(\varepsilon_{r,1} + \varepsilon_{r,2})}\, e_z, \qquad E_2 = -\frac{2\varepsilon_{r,1}\, U_0}{d(\varepsilon_{r,1} + \varepsilon_{r,2})}\, e_z$$

For negative U_0, the vectors reverse. **Check:** substituting the two field magnitudes into the half-gap drops gives exactly U_0.

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Task 8: Electrostatics

8b · Series connection

1. Every field line crosses dielectric 1 and then dielectric 2. The same free charge magnitude sits on each plate, while the voltage divides between the layers, so the capacitors are in series:

$$C_1 \text{ in series with } C_2$$



Series. One path from the top plate to the bottom plate through both layers: the same charge on each, the voltage shared.

2. The interface has no free surface charge, so the normal displacement (and therefore the series charge) is common to both. **Check:** a parallel model would incorrectly assign the full plate voltage to each half-thickness layer.

Task 8: Electrostatics

8c · Series capacitance

1. For a parallel-plate layer, $C = \varepsilon A/t$. Each layer has area a^2 and thickness $d/2$:

$$C_1 = \frac{2\varepsilon_0\,\varepsilon_{\mathrm{r},1}\,a^2}{d}, \qquad C_2 = \frac{2\varepsilon_0\,\varepsilon_{\mathrm{r},2}\,a^2}{d}$$

2. Series capacitances add by reciprocals:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$C = \frac{2\varepsilon_0\,a^2\,\varepsilon_{\mathrm{r},1}\,\varepsilon_{\mathrm{r},2}}{d(\varepsilon_{\mathrm{r},1} + \varepsilon_{\mathrm{r},2})}$$

Check: this is smaller than either layer capacitance, as a series equivalent must be.

Task 8: Electrostatics

8d · Free surface charge density

1. The bottom face of the top plate points outward from the conductor in the $-e_z$ direction. The free charge density on a conductor surface is $D \cdot n$ just outside the metal:

$$\sigma_0 = D \cdot (-e_z)$$

2. Since D also points downward for positive U_0, the dot product is positive:

$$\sigma_0 = \frac{2\varepsilon_0\,\varepsilon_{r,1}\,\varepsilon_{r,2}\,U_0}{d(\varepsilon_{r,1} + \varepsilon_{r,2})}$$

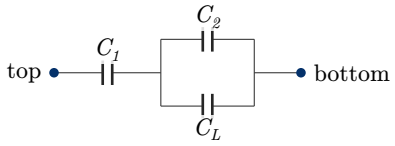
The units are C/m². Reversing the applied voltage reverses the surface-charge sign.

Task 8: Electrostatics

8e · Series and parallel sections

1. All paths pass through dielectric 1 across the full area a^2. In the lower half, some area $a(a-x)$ contains dielectric 2 and the remaining area $a\,x$ contains air. Those lower branches share the same voltage:

$$C_1 \text{ in series with } (C_2 \parallel C_L)$$



Series, then parallel. Every path crosses C_1; below it the path splits between the $\varepsilon_{r,2}$ part (C_2) and the air strip (C_L), which share one voltage.

2. Thus C_2 and C_L are in parallel with one another, and their equivalent is in series with C_1. **Check:** at $x = 0$ the air branch has zero area; at $x = a$ the lower dielectric branch has zero area.

Task 8: Electrostatics

8f · Capacitance after displacement

1. The upper layer keeps area a^2 and thickness $d/2$. The two lower branches each have thickness $d/2$:

$$C_1 = \frac{2\varepsilon_0\,\varepsilon_{\mathrm{r},1}\,a^2}{d}$$

$$C_2(x) = \frac{2\varepsilon_0\,\varepsilon_{\mathrm{r},2}\,a(a-x)}{d}, \qquad C_{\mathrm{L}}(x) = \frac{2\varepsilon_0\,a\,x}{d}$$

2. Add the parallel pair and then use the series formula:

$$C_{\mathrm{p}}(x) = C_2(x) + C_{\mathrm{L}}(x) = \frac{2\varepsilon_0\,a}{d}[\varepsilon_{\mathrm{r},2}\,a + (1-\varepsilon_{\mathrm{r},2})x]$$

$$C(x) = \frac{C_1\,C_{\mathrm{p}}(x)}{C_1 + C_{\mathrm{p}}(x)}$$

$$C(x) = \frac{2\varepsilon_0\,a}{d}\cdot\frac{\varepsilon_{\mathrm{r},1}\,a[\varepsilon_{\mathrm{r},2}\,a + (1-\varepsilon_{\mathrm{r},2})x]}{\varepsilon_{\mathrm{r},1}\,a + \varepsilon_{\mathrm{r},2}\,a + (1-\varepsilon_{\mathrm{r},2})x}$$

Endpoint checks: $C(0)$ matches 8c, while $C(a)$ is the series combination of the upper dielectric and an air layer of thickness $d/2$.

Task 8: Electrostatics

8g · Locate the extrema

1. The lower equivalent is linear in x:

$$C_{\mathrm{p}}(x) = \frac{2\varepsilon_0\,a}{d}[\varepsilon_{\mathrm{r},2}\,a + (1-\varepsilon_{\mathrm{r},2})x]$$

Because $\varepsilon_{\mathrm{r},2}$ is much greater than 1, its slope is negative. The series equivalent $C_1C_{\mathrm{p}}/(C_1+C_{\mathrm{p}})$ increases monotonically with C_{p}.

2. Therefore the total capacitance decreases throughout $0 \le x \le a$:

$$\text{maximum at } x = 0, \qquad \text{minimum at } x = a$$

If $\varepsilon_{\mathrm{r},2}$ were exactly 1, the lower region would be air everywhere and the capacitance would be constant; the sheet specifies $\varepsilon_{\mathrm{r},2}$ much greater than 1, excluding that degenerate case.

Personal worked-solution guide based on the supplied EE1 Tutorial Task sheet, written in my own words; no tutorial text or figures are reproduced. Not an official FAU publication; check course details against current material. Made by **Thanate Amornjiraporn (Focus)**.