Tutorial 5: Worked solutions

Electrical Engineering I · Module 92315 · Tutorial 5

From Fundamentals to AI Apocalypse · personal study guide

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Made by Thanate Amornjiraporn (Focus)

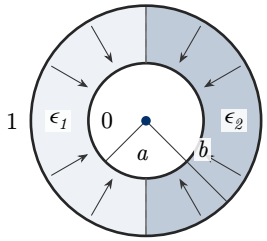
Use the coaxial symmetry to find the radial field, then apply the dielectric boundary conditions and integrate charge over the two half-cylinders.

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Task 9: Electrostatics

Two ideal conducting coaxial cylinders have radii a and b (with $0 < a < b$) and length l. The annular gap contains permittivity ε_1 for $90° < \varphi < 270°$ and ε_2 for $-90° < \varphi < 90°$. These are absolute permittivities. The conductors are held at potentials $\varphi_{e,1}$ on the outer cylinder and $\varphi_{e,0}$ on the inner; $U_{10} = \varphi_{e,1} - \varphi_{e,0}$. Neglect fringing at the finite ends.

Connected topics: Dielectric interfaces (§1.11) supplies the field boundary conditions; capacitance (§1.13) connects the resulting charge and voltage.



The field for $U_{10} > 0$. ε_1 fills the left half ($90° < \varphi < 270°$) and ε_2 the right half. E runs radially inward from the outer conductor 1 to the inner conductor 0, the same in both halves, and grows towards the inner cylinder as $1/\rho$; along the dashed interfaces it runs parallel to the boundary.

9a: Sketch the electric field

Try first: identify the direction perpendicular to the conducting cylinders and compare it with the normal to each dielectric interface.

Answers: see "Task 9: Electrostatics" under Answers at the end.

9b: Behaviour at the dielectric interface

Try first: at $\varphi = \pm 90°$, name the surface normal and resolve the radial field into normal and tangential components.

Answers: see "Task 9: Electrostatics" under Answers at the end.

9c: Mathematical approach for *E*

Try first: write Laplace's equation for a potential independent of φ and z, then use the two conductor potentials as boundary values.

Answers: see "Task 9: Electrostatics" under Answers at the end.

9d: Relationship between *D* and *E*

Try first: use the constitutive law separately in each angular material region.

Answers: see "Task 9: Electrostatics" under Answers at the end.

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9e: Sign of charge on the outer cylinder

Try first: compare the specified potential of the outer conductor with the inner one and use the direction of E.

Answers: see "Task 9: Electrostatics" under Answers at the end.

9f: Find E as a function of outer charge Q

Try first: relate the field scale to the charge on the outer conductor by integrating D over its inner cylindrical surface, split into two half-circumferences.

Answers: see "Task 9: Electrostatics" under Answers at the end.

9g: Find U_{10} as a function of Q

Try first: integrate the inward radial field between a and b, keeping the voltage reference $U_{10} = \varphi_{\mathrm{e,outer}} - \varphi_{\mathrm{e,inner}}$ straight.

Answers: see "Task 9: Electrostatics" under Answers at the end.

9h: Capacitance and numerical value

Try first: use $C = Q/U_{10}$, then substitute the two sector permittivities and dimensions.

Answers: see "Task 9: Electrostatics" under Answers at the end.

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Answers

Task 9: Electrostatics

9a · Qualitative field

1. The metal cylinders are equipotentials. Away from the finite ends, the field meets each cylindrical conductor normally, so its direction is radial, along e_ρ.
2. The voltage is $U_{10} = \varphi_{\mathrm{e,outer}} - \varphi_{\mathrm{e,inner}}$. If $U_{10} > 0$, the outer conductor is at higher potential, so E points inward ($-e_\rho$) from the outer cylinder toward the inner one. Reverse the arrows if $U_{10} < 0$.
3. Draw radial lines from radius b to radius a, with denser spacing near the inner cylinder because the field magnitude varies as $1/\rho$. Neglect end fringing as the ideal long-coax model requires.

Task 9: Electrostatics

9b · Boundary conditions

1. A constant-φ interface has normal $n = \pm e_\varphi$. The radial field is perpendicular to that normal, so it lies tangent to the interface:

$$E_{\mathrm{n},1} = E_{\mathrm{n},2} = 0, \qquad E_{\mathrm{t},1} = E_{\mathrm{t},2}$$

2. With no free charge on the dielectric boundary, the normal component of displacement is continuous; it is zero on both sides here. Tangential displacement need not be continuous and scales with each material permittivity:

$$\frac{D_{\mathrm{t},1}}{D_{\mathrm{t},2}} = \frac{\varepsilon_1}{\varepsilon_2}$$

3. Thus E crosses the angular boundary without a change in direction or magnitude. D changes magnitude at the boundary in proportion to ε. The radial field lines run along the interface, rather than crossing it.

Task 9: Electrostatics

9c · Radial solution

1. The coaxial geometry and constant conductor potentials allow a z- and φ-independent potential in each sector. The cylindrical Laplace equation reduces to:

$$\frac{1}{\rho} \cdot \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho \frac{\mathrm{d}\varphi_\mathrm{e}}{\mathrm{d}\rho}\right) = 0$$

$$\varphi_\mathrm{e}(\rho) = A \ln \rho + B$$

2. Apply $\varphi_\mathrm{e}(a) = \varphi_{\mathrm{e},0}$ and $\varphi_\mathrm{e}(b) = \varphi_{\mathrm{e},1}$. The same solution works in both sectors because their conductor boundary values are identical and the radial field is tangent to the interface:

$$\varphi_\mathrm{e}(\rho) = \varphi_{\mathrm{e},0} + \frac{U_{10}}{\ln(b/a)} \ln\left(\frac{\rho}{a}\right)$$

3. The electric field is the negative potential gradient:

$$E(\rho) = -\frac{U_{10}}{\rho \ln(b/a)}\, e_\rho$$

The $1/\rho$ dependence is required by cylindrical symmetry. Positive U_{10} gives the inward direction.

Task 9: Electrostatics

9d · Piecewise permittivity

1. For a linear, isotropic dielectric, $D = \varepsilon E$. The permittivity depends on which half-annulus contains the point:

$$D(\rho, \varphi) = \varepsilon_1 E(\rho) \qquad \text{for } 90° < \varphi < 270°$$

$$D(\rho, \varphi) = \varepsilon_2 E(\rho) \qquad \text{for } -90° < \varphi < 90°$$

2. Because E is the same on both sides of the radial interface, the displacement magnitudes differ by $\varepsilon_1/\varepsilon_2$. At $\varphi = \pm 90°$ the normal displacement remains zero and continuous.

Task 9: Electrostatics

9e · Outer-conductor charge sign

1. For $U_{10} > 0$, the outer cylinder's potential exceeds the inner cylinder's. The field points from the outer conductor toward the inner conductor.
2. The normal from the metal into the dielectric at the outer cylinder's inner wall is $-e_\rho$, parallel to E. The conductor surface charge density there is $D \cdot (-e_\rho) > 0$. Therefore the net charge on the outer cylinder is **positive**.

Task 9: Electrostatics

9f · Charge fixes the field scale

1. Let the inward field have magnitude K/ρ, so D in sector i has magnitude $\varepsilon_i K/\rho$. The normal from the outer metal into the gap is $-e_\rho$; its facing area is half the cylinder, $\pi b\, l$, in each sector:

$$Q = \frac{\varepsilon_1 K}{b}(\pi b l) + \frac{\varepsilon_2 K}{b}(\pi b l)$$

2. Solve for the field constant and substitute into the radial field:

$$K = \frac{Q}{\pi l(\varepsilon_1 + \varepsilon_2)}$$

$$E(\rho) = -\frac{Q}{\pi l(\varepsilon_1 + \varepsilon_2)\,\rho}\, e_\rho$$

This is uniform in φ although D is piecewise. The sign of Q sets the field direction.

Task 9: Electrostatics

9g · Charge–voltage relation

1. Because the field points inward, the outer-minus-inner voltage is $U_{10} = -\int_a^b E \cdot e_\rho \, \mathrm{d}\rho$:

$$U_{10} = K \int_a^b \frac{\mathrm{d}\rho}{\rho} = K \ln(b/a)$$

2. Insert K from 9f:

$$U_{10} = \frac{Q \ln(b/a)}{\pi l(\varepsilon_1 + \varepsilon_2)}$$

Positive Q gives positive U_{10}, consistent with 9e.

Task 9: Electrostatics

9h · Coaxial capacitance

1. For this linear arrangement, capacitance is the charge magnitude per applied voltage. Cancel Q using 9g:

$$C = \frac{\pi l(\varepsilon_1 + \varepsilon_2)}{\ln(b/a)}$$

2. With $\varepsilon_1 = \varepsilon_0$, $\varepsilon_2 = 3\varepsilon_0$, $a = 0.020$ m, $b = 0.021$ m and $l = 0.10$ m:

$$C = \frac{4\pi\varepsilon_0 \cdot 0.10\,\mathrm{m}}{\ln(0.021/0.020)}$$

$$C \approx 2.28 \times 10^{-10}\,\mathrm{F} = 228\,\mathrm{pF}$$

Check: a uniform ε_0 gap would have half this capacitance, since the mean permittivity here is $2\varepsilon_0$. The result is also positive and has units of farads.

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