Statements, Logic, and Sets

Mathematics for Engineers I · Module 62765 · Chapter 1

From Fundamentals to AI Apocalypse · personal study guide

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Mathematical arguments use a small grammar of statements, logical connectives, and sets. This chapter builds that grammar: first decide when statements are true, then describe and combine collections of objects precisely.

Start from memory

Try these before reading. If you are unsure, write a guess; the gap tells you what to look for.

1. When is an implication $A \Rightarrow B$ false?
2. Does mathematical "or" allow both statements to be true?
3. How do you negate "for every x, P(x)"?
4. What does $A \subset B$ say about the elements of A?
5. How is $A \times B$ different from the set containing A and B?

Answers: see "Start from memory" under Answers at the end.

How to use this guide

Read one section, hide the page, then reconstruct the rules and examples from memory. Use the answers only after you have committed to a response.

1. **Day 0:** Do the cold start, read §§1.1–1.2, and try the worked examples with the answers covered.
2. **Day 1:** Rebuild the truth table and the set operations from memory; solve practice 1–8.
3. **Day 3:** Solve practice 9–16 without notes, then check each answer and revisit its section reference.
4. **About day 7:** Solve practice 17–24 in mixed order and explain the De Morgan rules aloud.
5. **About day 14:** Solve practice 25–33, reconstruct the derivations, then take one exam problem under time limits.
6. **Before the exam:** Complete the self-check and both exam problems. Return to any skill you cannot demonstrate unaided.

When stuck: identify the definitions involved, test a small example, and write each membership or truth condition explicitly before simplifying.

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How the chapter fits together

The chapter moves from evaluating statements to describing collections and proving relationships between them.

Measured · the starting language

- **Statements** §1.1

  true = 1 · false = 0

  Rests on: deciding whether a sentence has a definite truth value.

- **Connectives** §1.1

  $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$

  Rests on: truth values for the component statements and the connective's rule.

Defined · objects and relations

- **Quantifiers** §1.1

  $\forall x\ P(x)$ · $\exists x\ P(x)$

  Rests on: a domain of possible objects and a predicate whose truth can be checked.

- **Sets and membership** §1.2

  $x \in A$

  Rests on: a well-defined collection and a membership test.

- **Set operations** §1.2

  $\cup$, $\cap$, $\setminus$, $\times$, A^c

  Rests on: membership in the original sets and, for a complement, a fixed universal set.

Derived · rules from definitions

- **Logical equivalences** §1.1

  truth tables prove equivalence

  Rests on: matching truth values in every row.

- **Set algebra** §1.2

  compare membership conditions

  Rests on: showing that both sides contain exactly the same elements.

Question	Tool
When is a compound statement true?	Truth table
Does a set equation hold?	Element-by-element membership argument
What is the negation of a quantified claim?	Swap $\forall$ and $\exists$, then negate the predicate

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1.1 Propositional Logic

Logic gives precise rules for combining claims. It lets you test arguments, simplify conditions, and express properties that will reappear throughout mathematics.

> **In plain words:** logic is a set of rules for combining claims that are either true or false. $\neg$ flips a claim, $\wedge$ (and) needs both parts true, $\vee$ (or) needs at least one, and $\Rightarrow$ (if … then) fails only when the first part is true and the second false. A **quantifier** says how many things a claim is about: $\forall$ means every one, $\exists$ means at least one.

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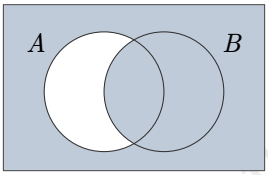
A statement has a truth value

A **statement** is a sentence that is either true or false; its truth value is 1 or 0, respectively. "$9 > 8$" is a statement. "Friday the 13th is unlucky" is not a statement here because it does not have a definite mathematical truth value. A claim may still be a statement even if its truth is not currently known.

Logical connectives

$$\neg A$$
$$A \wedge B$$
$$A \vee B$$
$$A \Rightarrow B$$
$$A \Leftrightarrow B$$

Read it as: "not A", "A and B", "A or B (or both)".
Truth tables define the meaning of each connective. The notes use 1 for true and 0 for false; $\vee$ is inclusive "or".



A $\Rightarrow$ **B** is false only where A is true and B is not (the white crescent); everywhere else it is true.

Form	Meaning	True when
¬A	not A	A is false
A ∧ B	A and B	both A and B are true
A ∨ B	A or B	at least one is true; both may be true
A ⇒ B	A implies B	A is false or B is true
A ⇔ B	A iff B	A and B have the same truth value

The implication A ⇒ B is false in exactly one case: A is true and B is false. This is why a false premise makes the implication true under the truth-table definition; it does not claim that the conclusion follows from a false premise in ordinary causal language. The notes' reason: nothing false can be deduced from something true, but anything can come from something false. From the false statement 2=1, multiplying both sides by 0 gives the true statement 0=0.

Take "if it is raining, the street is wet", with A = "it is raining" and B = "the street is wet". It is true when rain wets the street, when it is dry and the street is dry, and when it is dry and the street is wet from a hose. It is false only when it rains and the uncovered street stays dry.

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Worked example · build a truth table

Test whether $A \Rightarrow B$ has the same truth values as $\neg A \vee B$.

1. List all input pairs: (1,1), (1,0), (0,1), (0,0).
2. Evaluate $\neg A \vee B$: 1, 0, 1, 1.
3. Evaluate $A \Rightarrow B$: 1, 0, 1, 1.
4. The columns match in every row, so the two forms are equivalent, and $(A \Rightarrow B) \Leftrightarrow (\neg A \vee B)$ has the output 1, 1, 1, 1: it is a tautology.
5. Check: at A=1, B=0 the left side is false by definition, and $\neg A \vee B = 0 \vee 0 = 0$. §1.1

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Tautologies and logical rules

A **tautology** is true for every assignment of truth values. Two forms are logically equivalent when their truth values agree in every row. A truth table is a direct proof for expressions with a small number of variables: four rows for two variables, eight for three.

Rules to recognise and prove with truth tables

$$\neg\neg A \Leftrightarrow A$$
$$A \wedge B \Leftrightarrow B \wedge A$$
$$A \vee B \Leftrightarrow B \vee A$$
$$A \wedge (B \wedge C) \Leftrightarrow (A \wedge B) \wedge C$$
$$A \vee (B \vee C) \Leftrightarrow (A \vee B) \vee C$$
$$A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$$
$$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$$
$$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$$
$$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$$

These include double negation, commutativity, associativity, distributivity, and De Morgan's laws (Theorem 1.4, notes p. 4). Compare the commutative, associative and distributive rules with those for + and · in arithmetic. Unlike arithmetic, both distributive laws hold here.

Worked example · both De Morgan laws by table

Prove ¬(A ∧ B) ⇔ ¬A ∨ ¬B and ¬(A ∨ B) ⇔ ¬A ∧ ¬B.

A	**B**	**¬(A ∧ B)**	**¬A ∨ ¬B**	**¬(A ∨ B)**	**¬A ∧ ¬B**
1	1	0	0	0	0
1	0	1	1	0	0
0	1	1	1	0	0
0	0	1	1	1	1

1. Fill the third and fifth columns from the definitions of ∧ and ∨, then negate.
2. Fill the fourth and sixth columns from ¬A and ¬B.
3. Columns three and four match, and so do five and six, so both laws hold.
4. Check: A=B=0 makes "neither" true, and only the last row has a 1 in columns five and six. §1.1

Trap: the converse B ⇒ A is not generally equivalent to A ⇒ B. "If it rains, the street is wet" does not imply "if the street is wet, it rained." (The contrapositive ¬B ⇒ ¬A has the same truth table as A ⇒ B. This follows directly from the notes' truth table; the notes do not name it.)

Predicates and quantifiers

A **propositional form** $A(x_1,\ldots,x_n)$ depends on variables; assigning values turns it into a statement. For A(x,y): 5+x=10y, A(5,1) is true and A(6,1) is false. Below, P(x) is a form in one variable. A quantifier specifies how many values in a domain must satisfy it.

Quantifiers

$$\forall x\, P(x) : \ P(x) \text{ holds for every } x \text{ in the domain}$$
$$\exists x\, P(x) : \ P(x) \text{ holds for at least one } x$$
$$\exists! x\, P(x) : \ P(x) \text{ holds for exactly one } x$$

Read it as: "for all x, P of x" and "there exists an x with P of x".
The domain matters. Quantifier order matters too; see the dog example below.

Let D be a dog and P a person. ∃D ∀P: D bites P says one dog bites everyone. ∀P ∃D: D bites P says each person is bitten by some dog, possibly different dogs.

Negation changes "every" to "there exists at least one counterexample", and changes "there exists" to "none exist".

Negating quantified statements

$$\neg(\forall x\, P(x)) \Leftrightarrow \exists x\, \neg P(x)$$
$$\neg(\exists x\, P(x)) \Leftrightarrow \forall x\, \neg P(x)$$

Read it as: "not every x has P" says the same as "some x does not have P".
Negate the predicate after swapping the quantifier. The notes also write ∃ with a stroke through it for ¬∃.

Worked example · negate a quantified claim
Negate: "For every real number x, $x^2 \geq 0$."

1. Swap ∀ for ∃.
2. Negate the predicate $x^2 \geq 0$ to $x^2 < 0$.
3. The negation is "There exists a real number x with $x^2 < 0$." It is false, so the original claim is true.
4. Check: exactly one of a claim and its negation is true, and every real square is at least 0. §1.1

Worked example · negate nested quantifiers
Negate: ∃D ∀P: D bites P.

1. Work from the outside in: ¬∃D becomes ∀D ¬, leaving ¬(∀P: D bites P).
2. Then ¬∀P becomes ∃P ¬, which gives ∀D ∃P: ¬(D bites P).
3. In words: for every dog there is at least one person it does not bite.
4. Check: a dog that bites everyone would make the original true and this one false, as a negation must. §1.1

Memory hook: to disprove a universal claim, give one counterexample. To prove an existential claim, give one witness.

Applications: voting, circuits and SAT

These apply the truth rules; they add no new operators.

Three applications

$$V \Leftrightarrow (A \wedge B) \vee (A \wedge C) \vee (B \wedge C)$$
$$(A \wedge B) \vee (A \wedge \neg B) \Leftrightarrow A \wedge (B \vee \neg B) \Leftrightarrow A$$

The first is a safety vote over sensors A, B, C. The second simplifies a circuit by a distributive law.

- **Safety voting.** Aircraft and spacecraft use three independent sensors for one quantity. V is 1 when at least two report 1, so one faulty sensor cannot change V when the other two agree.
- **Circuits.** Logic gates realise ¬, ∧ and ∨. In hardware design the truth table is fixed in advance, and the task is the smallest circuit that achieves it. The simplification above replaces four gates by one wire.
- **SAT.** Satisfiability asks: given a formula $F(A_1,\ldots,A_n)$, is there an assignment of truth values that makes F true? A∧¬B is satisfiable, with A=1, B=0. A∧¬A is not, since no assignment makes it true. SAT solvers test and verify hardware and model constraints in scheduling and planning.

Near miss: "if n is divisible by 4, then n is even" is true, and its converse "if n is even, then n is divisible by 4" sounds just as reasonable, but it fails at $n = 2$. One counterexample is enough to sink a universal claim.

Recall check · answer without looking

1. Which row makes A ⇒ B false?
2. How can you prove two forms equivalent with a truth table?
3. What is the negation of ∀x P(x)?

Answers: see "1.1 Propositional Logic" under Answers at the end.

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1.2 Set Theory

Sets let us collect objects by a precise membership rule. Logical connectives reappear here as operations on membership conditions.

> **In plain words:** a set is a collection with a clear membership test: for any object you can say whether it is in or out. The **union** A ∪ B collects everything in either set, the **intersection** A ∩ B keeps what is in both, and the **difference** A \ B keeps what is in A but not in B. Each operation is a logic connective applied to the membership test.

Membership, descriptions, and inclusion

A **set** is a well-defined collection of distinct objects. Write $x \in A$ when x is a member and $x \notin A$ otherwise. A set may be given by listing its elements, such as {1,2,3}, or by a property, such as $\{x \in \mathbb{R} : x > 0\}$. The notes also write $\{x \mid A(x)\}$ for $\{x : A(x)\}$. The solutions of $x^3-2x=0$ form $\{x : x^3=2x\}=\{0,\sqrt{2},-\sqrt{2}\}$. Order and repetition do not change a set.

Subset and equality

$$A \subset B \Leftrightarrow \text{every } x \in A \text{ also belongs to } B$$
$$A = B \Leftrightarrow A \subset B \text{ and } B \subset A$$
$$A \subsetneq B \Leftrightarrow A \subset B \text{ and some element of } B \text{ is not in } A$$

Read it as: "A is a subset of B when every x in A is also in B"; two sets are equal when each is a subset of the other.

The notes use ⊂ for subset and ⊊ for proper subset. Some authors write ⊆ for subset and ⊂ for proper subset, so check the convention.

The empty set $\varnothing=\{x : x \neq x\}$ has no elements and is a subset of every set. Every set is also a subset of itself: $X \subset X$. Standard number sets are $\mathbb{N}$ (positive natural numbers in these notes), $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rationals), $\mathbb{R}$ (reals), and $\mathbb{C}$ (complex numbers).

Intervals, power sets, and size

Intervals describe parts of the real line. Round endpoints are excluded and square endpoints included: (a,b), [a,b), (a,b], [a,b]. Unbounded intervals include (a,∞) and $(-\infty,b]$; the notes also use $\mathbb{R}=(-\infty,\infty)$, $\mathbb{R}^+=(0,\infty)$, and $\mathbb{R}_{\geq 0}=[0,\infty)$. The power set $\mathcal{P}(X)$ contains every subset of X. For finite X, cardinality $|X|$, also written #X, counts its elements, and $|\mathcal{P}(X)| = 2^{|X|}$.

> **Worked example · enumerate a power set**
> Let X={a,b}. List $\mathcal{P}(X)$ and its size.
>
> 1. Include the empty subset: ∅.
> 2. Include the two one-element subsets: {a}, {b}.
> 3. Include the whole set: {a,b}. Therefore $\mathcal{P}(X)=\{\varnothing,\{a\},\{b\},\{a,b\}\}$ and $|\mathcal{P}(X)|=4$.
> 4. Check: $2^{|X|}=2^2=4$ agrees with the list, which contains ∅ and X. §1.2

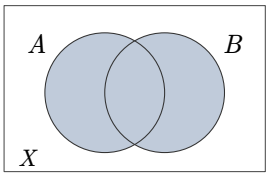
Set operations

Each operation is easiest to understand by asking when an arbitrary element x belongs to the result. Sets with empty intersection are called **disjoint**.

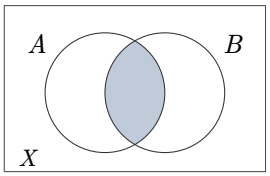
Operations and membership tests

$$x \in A \cup B \Leftrightarrow (x \in A) \vee (x \in B)$$
$$x \in A \cap B \Leftrightarrow (x \in A) \wedge (x \in B)$$
$$x \in A$$
$$, B \Leftrightarrow (x \in A) \wedge (x \notin B)$$
$$A \times B = \{(a, b) : a \in A \,\text{and}\, b \in B\}$$
$$A^c = X$$
$$, A, \text{ where } X \text{ is the fixed non} - \text{empty universal set}$$

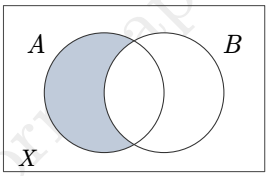
Read it as: "x is in A union B exactly when x is in A or x is in B", and the same with "and" for the intersection. Pairs are ordered: (a,b) is generally different from (b,a). A complement is defined for A ⊂ X, relative to X. The notes also write an overline, $\overline{A}$, for A^c.



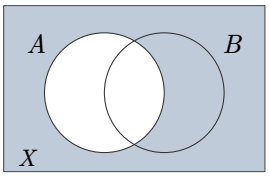
A ∪ B: everything in A, in B, or in both.



A ∩ B: only what is in both.



A \ B: what is in A but not in B.



A^c: everything in X that is not in A.

Union collects elements in either set; intersection keeps elements in both; difference keeps elements in A but removes those in B. The Cartesian product contains ordered pairs, not sets of the factors.

Operations on n sets

$$\bigcup_{k=1}^{n} A_k \text{ contains } x \text{ when } x \text{ belongs to at least one } A_k$$
$$\bigcap_{k=1}^{n} A_k \text{ contains } x \text{ when } x \text{ belongs to every } A_k$$
$$\prod_{k=1}^{n} A_k = A_1 \times \cdots \times A_n, \text{ the ordered } n - \text{tuples } (x_1, ..., x_n) \text{ with } x_k \in A_k$$

In particular, $\mathbb{R}^2$=$\mathbb{R}\times\mathbb{R}$ is the plane and $\mathbb{R}^3$=$\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ is three-dimensional space. Also, [1,3]×[1,2]={(x,y)∈$\mathbb{R}^2$: 1≤x≤3, 1≤y≤2} is a rectangle, and $\{0,1\}^3$={0,1}×{0,1}×{0,1} is the eight vertices of the unit cube.

Worked example · combine operations

Let $A=\{1,2,3\}$ and $B=\{3,4,5\}$. Find $A\cup B$, $A\cap B$, and $A\setminus B$.

1. Membership in at least one gives $A\cup B=\{1,2,3,4,5\}$.
2. Membership in both gives $A\cap B=\{3\}$, so the sets are not disjoint.
3. Members of A that are not in B give $A\setminus B=\{1,2\}$.
4. Check: re-test each element. 3 is in both, so it is in $A\cap B$ and not in $A\setminus B$; 4 is only in B, so it is in $A\cup B$ and not in $A\setminus B$. §1.2

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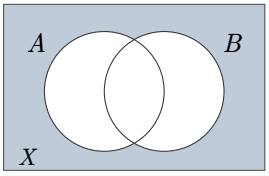
Set algebra and De Morgan's laws

Set identities can be proved by translating both sides into membership conditions. For example, $x \in (A \cap B)^c$ exactly when $x \notin A \cap B$; this means it is not true that x is in both sets, or equivalently $x \notin A$ or $x \notin B$. Thus $x \in A^c \cup B^c$, proving $(A \cap B)^c = A^c \cup B^c$.

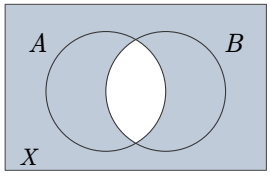
Set-algebra rules (Theorem 1.13)

$$
\begin{aligned}
(A^c)^c &= A \\
A \cap B &= B \cap A \\
A \cup B &= B \cup A \\
A \cap (B \cap C) &= (A \cap B) \cap C \\
A \cup (B \cup C) &= (A \cup B) \cup C \\
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\
(A \cap B)^c &= A^c \cup B^c \\
(A \cup B)^c &= A^c \cap B^c
\end{aligned}
$$

Complements use the same fixed universal set X.



$(\mathbf{A} \cup \mathbf{B})^c = \mathbf{A}^c \cap \mathbf{B}^c$: outside both circles, the same region either way.



$(\mathbf{A} \cap \mathbf{B})^c = \mathbf{A}^c \cup \mathbf{B}^c$: everything except the overlap.

Logic and sets side by side

$$
\begin{aligned}
A \wedge B &\leftrightarrow A \cap B \\
A \vee B &\leftrightarrow A \cup B \\
\neg A &\leftrightarrow A^c \\
A \Rightarrow B &\leftrightarrow A \subset B \\
A \Leftrightarrow B &\leftrightarrow A = B
\end{aligned}
$$

Each set rule in Theorem 1.13 is the matching logic rule of Theorem 1.4 applied to membership: for example $x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$. The notes prove (R1) this way: $x \in (A^c)^c \Leftrightarrow \neg(x \in A^c) \Leftrightarrow \neg\neg(x \in A) \Leftrightarrow x \in A$ (notes p. 11).

Trap: $A \setminus B$ is not generally $B \setminus A$. Check the direction by testing membership: the first set supplies candidates; the second removes them.

Near miss: $\{1, 2\} \subset \{1, 2, 3\}$ is true, but $\{1, 4\} \subset \{1, 2, 3\}$ fails: the single element 4 lies outside, and one element outside is enough.

Recall check · answer without looking

1. How do you prove A=B from the definition?
2. What is the difference between A∩B and A×B?
3. What information is needed before A^c is defined?

Answers: see "1.2 Set Theory" under Answers at the end.

Formula map and notation

Idea	Rule	Check
Implication	$A \Rightarrow B \Leftrightarrow \neg A \vee B$	False only for A=true, B=false
Negated quantifier	$\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)$	One counterexample is enough
De Morgan	$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$; $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$; $(A \cap B)^c = A^c \cup B^c$; $(A \cup B)^c = A^c \cap B^c$	Negate each part and swap the operator
Unique existence	$\exists! x\, P(x)$	Exactly one x: count them
Subset	$A \subset B \Leftrightarrow \forall x\, (x \in A \Rightarrow x \in B)$	$\varnothing \subset A$ and $A \subset A$
Equality	$A = B \Leftrightarrow A \subset B$ and $B \subset A$	Prove both directions
Disjoint	$A \cap B = \varnothing$	No shared element
Union/intersection	∪ means or; ∩ means and	Translate to membership
Difference	$A \setminus B$: in A and not in B	Order matters
Cartesian product	$A \times B = \{(a,b): a \in A \text{ and } b \in B\}$	Pairs are ordered
Complement	$A^c = X \setminus A$	State X
Power set	$\lvert\mathcal{P}(X)\rvert = 2^{\lvert X\rvert}$, finite X	Includes ∅ and X

Check every answer: for logic, test all truth assignments; for sets, test membership element by element and state the universal set when using complements.

Derive it cold

Rebuild these on blank paper before opening the answers in the relevant sections.

1. Use the truth table to derive $A \Rightarrow B \Leftrightarrow \neg A \vee B$. §1.1
2. Use the truth table to derive both logical De Morgan laws. §1.1
3. Negate a universal and an existential statement by swapping quantifiers and negating the predicate. §1.1
4. Prove one set identity by writing an arbitrary x-membership condition for each side. §1.2
5. Derive the set De Morgan laws from the logical De Morgan laws and membership tests. §1.2

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Explain it

Answer each "why" in your own words before checking the model response.

1. Why is A⇒B true when A is false under the truth-table definition?
2. Why does matching truth-table columns prove equivalence?
3. Why can one counterexample disprove a universal statement?
4. Why does quantifier order change the meaning of a claim about dogs and people?
5. Why must the universal set be fixed to define a complement?
6. Why is A\B different from B\A?
7. Why are ordered pairs in A×B generally not the same as those in B×A?
8. Why can a set identity be proved by comparing membership conditions?

Answers: see "Explain it" under Answers at the end.

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Mixed practice

Try all 33 problems before opening the answer key. The sections alternate so you must identify the right tool.

1. For A=1 and B=0, evaluate ¬A, A∧B, A∨B, A⇒B, and A⇔B.
2. Let A={1,2,3}, B={3,4}. Find A∪B, A∩B, A\B, and B\A.
3. Is "There are infinitely many twin primes" a statement even if its truth is unproven? Explain briefly.
4. Let X={p,q,r}. List $\mathcal{P}(X)$ and state its cardinality.
5. Use a truth table to verify ¬(A⇒B) ⇔ A∧¬B. Which input row makes both sides true?
6. Let A=[0,2] and B=(1,3). Find A∩B and A∪B.
7. Negate: "For every integer n, $n^2 \geq 0$."
8. Let X={1,2,3,4} and A={1,4}. Find A^c relative to X.
9. Build a truth table for ¬(A∨B). In which row is it true?
10. If |X|=3, how many elements does $\mathcal{P}(X)$ have?
11. Translate "There is exactly one real x such that $x^2=4$" using quantifier notation.
12. Let A={1,2}, B={x,y,z}. List A×B.
13. Are A⇒B and B⇒A equivalent for all A,B? Give one truth assignment that disproves equivalence.
14. Let X=ℝ and A=(−1,3]. Describe A^c as intervals.
15. Let P(x) mean "x is even", with domain ℤ. Write "some integer is not even" symbolically.
16. Let A={1,2,3}, B={2,3,4}, C={3,4,5}. Find A∩(B∪C).
17. Classify each as a statement or not: (i) 4 is prime; (ii) x+1=2; (iii) 4 is lucky.
18. Use De Morgan's law to simplify $(A\cap B)^c$.
19. For A=0 and B=0, compare A⇒B with ¬A∨B. Then use a truth table to verify ¬(A∧B) ⇔ ¬A∨¬B.
20. Let A={1,2,3,4} and B={2,4}. Is B⊂A? Is B⊊A?
21. Negate: "There exists a real x such that $x^2<0$."
22. Let X={1,2,3,4,5}, A={1,2,3}, B={3,4}. Find $(A\cup B)^c$ relative to X.
23. Let S be a student and T a tutor. Compare ∃T ∀S: T helps S with ∀S ∃T: T helps S in plain language.
24. For A={1,2} and B={2,3}, find A×B and B×A. State one ordered pair in only one of them.
25. Simplify (A∨B)∧(A∨¬B) with the logical rules, naming each rule. How many gates does this remove?
26. Let A={1,3}, B={2,4}, C={3,4}. Which pairs of these sets are disjoint?
27. For the sensor vote V, find V for A=B=1 with C=0 and with C=1. Then repeat for A=1, B=0. What do the results show about a faulty third sensor?
28. Let A_1={1,2}, A_2={2,3}, A_3={2,4}. Find $\bigcup_{k=1}^{3} A_k$ and $\bigcap_{k=1}^{3} A_k$. How many elements has $\prod_{k=1}^{3} A_k$? Give one.
29. Negate ∀S ∃T: T helps S, and say it in words.
30. Describe [0,2]×[1,3] as a set in $\mathbb{R}^2$ and name its shape. Is (2,3) in it? Is (3,2)?

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31. Use an 8-row truth table to prove $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$.
32. List every element of $\{0,1\}^3$. What do they represent geometrically?
33. Is $(A \vee B) \wedge \neg A$ satisfiable? Is $(A \vee B) \wedge \neg A \wedge \neg B$? Give an assignment or a reason.

Answers: see "Mixed practice" under Answers at the end.

Exam-style problems

Set a timer for about 20 minutes per problem. Write the reasoning, not just the final values.

1 · Logic for a sensor check

Three sensors report truth values A, B, and C. A majority vote is represented by $V=(A \wedge B) \vee (A \wedge C) \vee (B \wedge C)$.

1. Make a truth table for V and identify the rows where $V=1$.
2. Show that swapping any two sensor labels leaves V unchanged, using commutativity of $\wedge$ and $\vee$.
3. Negate the claim "For every sensor configuration with at most one report equal to 1, the majority output is 0." State the form of a counterexample.
4. Let S be the set of configurations (A,B,C) in $X=\{0,1\}^3$ with $V=1$. List S, then find $|S|$ and $|S^c|$ relative to X.

Answers: see "Exam-style problems" under Answers at the end.

2 · Sets of admissible inputs

Let the universal set be $X=\{1,2,3,4,5,6,7,8\}$. Let A be the even elements of X, $B=\{1,2,3,4\}$, and $C=\{3,4,5,6\}$.

1. List A, $A \cap B$, and $A \cup B$.
2. Find $(B \cup C)^c$ and $B^c \cap C^c$ relative to X, and use the results to check a De Morgan identity.
3. Let $D=\{\text{red, blue}\}$. List $A \times D$ and give its cardinality.
4. Write "every element of $A \cap B$ is even" with a quantifier, negate it, and say which of the two is true.

Answers: see "Exam-style problems" under Answers at the end.

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Slips in the notes

These are the places where the notes are wrong, loose or need a qualification, and where this guide differs from them. Printed page numbers refer to the notes.

1. **Printed page 3, error:** the example gives $\neg(9 \geq 8)$ as "$9 \leq 8$". The negation of $\geq$ is $<$, so it should be $9 < 8$. The printed form happens to be false too, but it is not the negation: $8 \leq 8$ and $8 \geq 8$ both hold.
2. **Printed page 4, error:** the two coffee sentences are not equivalent. "Milk or sugar, then I do not drink" is $(M \vee S) \Rightarrow \neg D$, which is equivalent to its contrapositive $D \Rightarrow (\neg M \wedge \neg S)$. The printed second sentence, $(\neg M \wedge \neg S) \Rightarrow D$, is the converse of that. They differ for $M=S=D=0$, where the first is true and the second false. R9 alone only rewrites the premise. This guide leaves the example out.
3. **Printed page 9, edge case:** the notes say that $A \times B$ is neither contained in A nor in B. If either factor is empty, then $A \times B = \varnothing$, and $\varnothing$ is a subset of both factors. The definition of the Cartesian product remains correct; the remark about containment is too broad for empty factors.
4. **Printed page 5, minor:** "x is divisible only by 1 and itself" is glossed as "x is prime", but $x=1$ satisfies the formula and is not prime. The quantified statement stays true.
5. **Printed page 6, loose wording:** "a single faulty sensor cannot change the truth value" holds only when the other two sensors agree, which is how the page ends ("two sensors must agree") and how §1.1 here states it. If they disagree, the third decides V.
6. **Printed page 6, notation:** the remark on the crossed-out $\exists$ joins statements with "=". This guide writes $\Leftrightarrow$ between statements.
7. **Printed page 7, minor:** Definition 1.8(iii) writes $A = B :\Leftrightarrow (x \in A \Leftrightarrow x \in B)$ without the $\forall x$ that part (i) has. This guide uses the wording of that definition, $A \subset B$ and $B \subset A$, which carries the quantifier.
8. **Printed page 3, typographic:** leftover German ("Dann gilt für die neuen Aussagen") stands before the English sentence. It has no effect on the content.

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Self-check

Use day 0 to learn and practise. Tick a skill only in a later session, at least one day afterward, when you can show it without notes. Use the listed problem numbers as evidence.

- ☐ Evaluate all five logical connectives from input truth values. (1, 5, 9, 13, 19, 27)
- ☐ Distinguish a statement from an open or undefined sentence. (3, 17)
- ☐ Prove equivalence with complete truth tables, simplify, and test satisfiability. (5, 19, 25, 31, 33)
- ☐ Translate and negate quantified claims. (7, 11, 15, 21, 23, 29)
- ☐ Compute unions, intersections, and differences from membership, for two or n sets. (2, 6, 16, 26, 28)
- ☐ Use subset, proper subset, and equality definitions. (20)
- ☐ Find complements relative to an explicit universe. (8, 14, 22)
- ☐ Enumerate small power sets and count subsets. (4, 10)
- ☐ List Cartesian products and track coordinate order. (12, 24, 30, 32)
- ☐ Apply De Morgan's laws to statements and sets. (18, 19)
- ☐ Rebuild the listed derivations from definitions.
- ☐ Solve both multi-part problems and justify each step.

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Answers

Start from memory

1. Only when A is true and B is false. §1.1
2. Yes. $A \vee B$ is true when at least one is true, including when both are true. §1.1
3. There exists an x for which P(x) is false: $\exists x\, \neg P(x)$. §1.1
4. Every element of A is also an element of B. §1.2
5. $A \times B$ contains ordered pairs (a,b) with $a \in A$ and $b \in B$; it is not the set {A,B}. §1.2

1.1 Propositional Logic

1. A=1, B=0. §1.1
2. Show that their output columns agree for every assignment. §1.1
3. $\exists x\, \neg P(x)$: at least one value fails the predicate. §1.1

1.2 Set Theory

1. Show both $A \subset B$ and $B \subset A$. §1.2
2. $A \cap B$ is a set of elements common to both; $A \times B$ is a set of ordered pairs with first and second components in the respective sets. §1.2
3. A universal set X containing A. §1.2

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Explain it

1. An implication only forbids going from true to false. From a false premise you may reach anything: from 2=1, multiplying by 0 gives the true 0=0. So no row with A false can break it. §1.1
2. Equivalence means the same output for every possible input. The table lists every input assignment, so matching columns leave no case unchecked. §1.1
3. A universal claim says every domain element works. One failing element contradicts "every", so it is enough to make the claim false, however many others pass. §1.1
4. In $\exists D \forall P$ the dog is chosen first, so one dog must bite all people. In $\forall P \exists D$ the person comes first, so each person may have a different dog. The second can hold when the first fails. §1.1
5. "Outside A" means outside A but still within the chosen universal set. Change X and different elements count as outside, so the complement changes. §1.2
6. $A \setminus B$ starts from A and removes elements of B. Reversing the order changes both the candidates and what is removed, so $\{1,2\} \setminus \{2,3\} = \{1\}$ but $\{2,3\} \setminus \{1,2\} = \{3\}$. §1.2
7. The first coordinate must come from A and the second from B. Swapping the factors swaps those roles, and since $(1,2) \neq (2,1)$ the two products generally hold different pairs. §1.2
8. Two sets are equal when they contain exactly the same elements. An arbitrary x stands for every possible element, so matching membership conditions cover them all. §1.2

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Mixed practice

1. With A=1 and B=0, ¬A=0, A∧B=0, A∨B=1, A⇒B=0, and A⇔B=0. The implication fails only on this true-to-false row; equivalence fails because the inputs differ. §1.1
2. Members in either set give A∪B={1,2,3,4}; the shared member gives A∩B={3}. Removing B from A leaves {1,2}, while removing A from B leaves {4}. §1.2
3. Yes. It is a declarative claim with a definite truth value, although that value is not known. §1.1
4. $\mathcal{P}(X)=\{\varnothing,\{p\},\{q\},\{r\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$ and $|\mathcal{P}(X)|=8=2^3$. §1.2
5. For rows (A,B)=(1,1),(1,0),(0,1),(0,0), A⇒B is 1,0,1,1, so ¬(A⇒B) is 0,1,0,0. Also A∧¬B is 0,1,0,0. The columns agree, and both are true only for A=1,B=0. §1.1
6. A∩B=(1,2]; A∪B=[0,3). §1.2
7. There exists an integer n with $n^2<0$. This is false over $\mathbb{Z}$. §1.1
8. $A^c=X\setminus A=\{1,2,3,4\}\setminus\{1,4\}=\{2,3\}$; the complement is taken within X. §1.2
9. It is true only when A=0 and B=0, because both must be false for ¬(A∨B). §1.1
10. A three-element set has $2^3=8$ subsets: each element is either included or excluded independently. §1.2
11. $\exists!x\in\mathbb{R}$ such that $x^2=4$. The solutions are 2 and −2, so the claim as written is false. §1.1
12. {(1,x),(1,y),(1,z),(2,x),(2,y),(2,z)}. The first coordinate comes from A. §1.2
13. No. With A=1,B=0, A⇒B is false but B⇒A is true. §1.1
14. $(-\infty,-1]\cup(3,\infty)$. The point −1 is outside A, so it joins the complement; 3 is in A, so it does not. §1.2
15. $\exists x\in\mathbb{Z}$ such that ¬P(x), or "there is an odd integer." §1.1
16. B∪C={2,3,4,5}; intersecting with A gives {2,3}. §1.2
17. (i) Statement, false. (ii) Not a statement until x is assigned or quantified. (iii) Not a mathematical statement because "lucky" has no defined truth condition. §1.1
18. $(A\cap B)^c=A^c\cup B^c$: being outside both together means being outside at least one. §1.2
19. For A=B=0, both A⇒B and ¬A∨B equal 1. For rows (1,1),(1,0),(0,1),(0,0), both ¬(A∧B) and ¬A∨¬B have values 0,1,1,1, so the columns match. §1.1
20. Yes, every element of B is in A. Yes, since $1\in A$ but $1\notin B$, so A has an element outside B. §1.2
21. For every real x, $x^2\geq 0$. §1.1
22. A∪B={1,2,3,4}; its complement in X is {5}. §1.2
23. The first says one tutor helps every student. The second says each student has at least one tutor who helps, possibly different tutors. §1.1
24. A×B={(1,2),(1,3),(2,2),(2,3)}; B×A={(2,1),(2,2),(3,1),(3,2)}. For example, (1,2) is only in A×B. §1.2
25. By the distributive law (R7), (A∨B)∧(A∨¬B) ⇔ A∨(B∧¬B). Since B∧¬B is always false, this is A. Two ORs, one AND and one NOT are replaced by a wire, so four gates go. §1.1
26. A and B share nothing, so A∩B=∅ and they are disjoint. A∩C={3} and B∩C={4}, so those pairs are not. §1.2
27. V=1 in both cases for A=B=1, because A∧B is already 1. For A=1, B=0, V=0 when C=0 and V=1 when C=1. A fault in C cannot change V when A and B agree, but C decides V when they disagree. §1.1

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28. The union is {1,2,3,4}; the intersection is {2}, the only element in all three. The product has 2·2·2=8 ordered triples, for example (1,3,4). §1.2

29. ∃S ∀T: ¬(T helps S). There is a student whom no tutor helps. §1.1

30. $\{(x,y)\in\mathbb{R}^2 : 0\le x\le 2,\ 1\le y\le 3\}$, a rectangle (here a square). (2,3) is in it, because both intervals are closed. (3,2) is not, because $3\notin[0,2]$. §1.2

31. For ABC=111,110,101,100,011,010,001,000: B∨C is 1,1,1,0,1,1,1,0, so the left side is 1,1,1,0,0,0,0,0. A∧B is 1,1,0,0,0,0,0,0 and A∧C is 1,0,1,0,0,0,0,0, so the right side is 1,1,1,0,0,0,0,0. The columns match. §1.1

32. (0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1): $2^3=8$ triples, the vertices of the unit cube. §1.2

33. (A∨B)∧¬A is satisfiable, with A=0, B=1. (A∨B)∧¬A∧¬B is not: by De Morgan ¬A∧¬B ⇔ ¬(A∨B), so the formula has the form F∧¬F. §1.1

Exam-style problems

1. V=1 when at least two reports are 1: rows 111, 110, 101, 011. It is 0 for 100, 010, 001, 000. §1.1

2. For example, swapping A and B changes (A∧B) to (B∧A), which is equal by commutativity; it swaps the other two terms as well. The full disjunction is unchanged, and the same argument applies to any pair. §1.1

3. ∃ a configuration with at most one 1 but V=1. This cannot happen for the majority rule, so the original claim is true. §1.1

4. From part 1, S={(1,1,1),(1,1,0),(1,0,1),(0,1,1)}, so |S|=4. X has $2^3=8$ elements, so $|S^c|=8-4=4$. §1.1 §1.2

Exam-style problems

1. A={2,4,6,8}; A∩B={2,4}; A∪B={1,2,3,4,6,8}. §1.2

2. B∪C={1,2,3,4,5,6}, so $(B\cup C)^c=\{7,8\}$. Also $B^c=\{5,6,7,8\}$ and $C^c=\{1,2,7,8\}$; their intersection is {7,8}. Hence $(B\cup C)^c=B^c\cap C^c$. §1.2

3. A×D={(2,red),(2,blue),(4,red),(4,blue),(6,red),(6,blue),(8,red),(8,blue)}; it has 8 ordered pairs. §1.2

4. ∀x∈A∩B: x is even. The negation is ∃x∈A∩B: x is not even. Since A∩B={2,4}, the first is true and the negation false. §1.1 §1.2

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