Real Numbers

Mathematics for Engineers I · Module 62765 · Chapter 2

From Fundamentals to AI Apocalypse · personal study guide

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This chapter starts with number systems, builds the real numbers from algebra, order and completeness, then uses those properties to solve inequalities and prove identities. The final tools—binomial coefficients and induction—turn repeated patterns into general results.

Start from memory

Try these before reading. Write an answer even if you are uncertain; the gap tells you what to look for.

1. Which number systems sit between the natural numbers and the real numbers?
2. What extra property distinguishes the real numbers from the rationals?
3. What does $|x-a|$ measure on the number line?
4. How many ways can you choose k objects from n when order does not matter?
5. What are the two essential steps in an induction proof?

Answers: see "Start from memory" under Answers at the end.

How to use this guide

Attempt before reading each worked solution. Then hide the guide and retrieve the method on a later day.

1. **Day 0:** Do the cold start. Read §§2.1–2.2 and reconstruct the number-system map and the definitions of supremum and completeness.
2. **Day 1:** Read §§2.3–2.4. Re-solve their worked examples from a blank page, then try mixed practice 1–8.
3. **Day 3:** Study §2.5 and the derivation list. Solve practice 9–16 without notes; check each answer and revisit its cited section.
4. **About day 7:** Shuffle practice 17–25. Explain aloud why each method applies, then try one exam problem timed.
5. **About day 14:** Rebuild all derivations and retry every marked question. Complete the self-check only with evidence from unaided work.
6. **Before the exam:** Do both exam problems under time limits, then review only the skills your evidence shows are weak.

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How the chapter fits together

Number systems lead to the complete ordered field $\mathbb{R}$. Its structure supports inequalities, counting formulas and induction.

Measured · starting objects

- **Number systems and binary** §2.1
  $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$; binary uses 2^k
  Rests on: set membership and place value.

Defined · structure and tools

- **Complete ordered field** §2.2
  $\mathbb{R}$: algebra + order + suprema
  Rests on: field, order and least upper bounds.
- **Absolute value** §2.3
  $|x-a|$ = distance to a
  Rests on: order cases and additive inverses.
- **Binomial coefficient** §2.4
  $C(n,k) = n!/[k!(n-k)!]$
  Rests on: counting selections without order.

Derived · reusable results

- **Triangle inequality** §2.3
  $|x+y| \leq |x|+|y|$
  Rests on: absolute value and order rules.
- **Binomial theorem** §2.4
  $(a+b)^n = \sum_{k=0}^{n} C(n,k)a^{n-k}b^k$
  Rests on: factorial algebra and Pascal's identity.
- **Induction** §2.5
  $A(n_0)$ and $A(n) \Rightarrow A(n+1)$
  Rests on: a start and a valid next-step implication.

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Choose a tool

If the task asks...	Choose this tool
Which number set contains a value?	Definitions and set inclusions.
What is the least upper bound?	Bounds and completeness.
Which values satisfy a distance condition?	Absolute value as an interval or cases.
How many terms or choices?	Binomial theorem or C(n,k).
Must a claim hold for every n?	Induction with a base and step.

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2.1 Number Systems

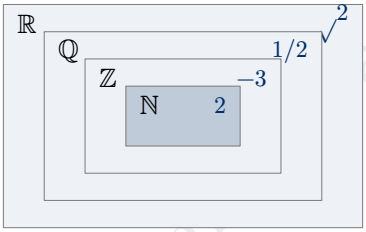
The sets grow by solving equations that smaller number systems cannot always solve. Knowing which set contains a number determines which tools are valid.

> **In plain words:** each number system fixes an equation the smaller one cannot always solve. In $\mathbb{N}$ you can count but not always subtract; $\mathbb{Z}$ adds the negatives, so $x + 3 = 1$ has a solution; $\mathbb{Q}$ adds fractions, so $2x = 1$ does; $\mathbb{R}$ fills the remaining gaps, so $x^2 = 2$ does. Each set sits inside the next.

Number-system inclusion

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$$
$$\text{irrational numbers} = \mathbb{R}$$
$$mathbbQ$$

Read it as: "$\mathbb{N}$ is a proper subset of $\mathbb{Z}$, which is a proper subset of $\mathbb{Q}$, which is a proper subset of $\mathbb{R}$": each system strictly contains the one before.
Here $\mathbb{N}=\{1,2,\ldots\}$; $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. A rational is p/q with $p\in\mathbb{Z}$, $q\in\mathbb{N}$ and p,q coprime.



Each number system sits inside the next, with an example that first appears at each level: $2 \in \mathbb{N}$, $-3 \in \mathbb{Z}$, $1/2 \in \mathbb{Q}$, $\sqrt{2} \in \mathbb{R}$.

Integers add negatives to $\mathbb{N}$. Rationals add quotients with non-zero integer denominator. The real numbers add limits such as $\sqrt{2}$ that cannot be expressed as a rational quotient. For example, $\sqrt{2}$ is irrational: if $\sqrt{2}=p/q$ in lowest terms, then $p^2=2q^2$ makes p even; substituting $p=2m$ makes q even too, contradicting lowest terms.

Binary uses place values that are powers of two. A bit string $[a_m\ldots a_0]_2$ represents $\sum_{k=0}^{m} a_k 2^k$, where each a_k is 0 or 1. For a fixed-point example, $[10.11]_2=2+1/2+1/4=2.75$. The field $\mathbb{Z}_2$ uses arithmetic modulo 2; ordinary binary integer addition and multiplication instead represent integer arithmetic and do not make the set of finite bit strings a field.

Worked example · classify and convert

1. $\sqrt{49}=7$, so it is natural, integer, rational and real.
2. $-7/3$ is rational and real, but not an integer.
3. $\sqrt{2}$ is real and irrational, so it is not rational.

$11010_2=1\cdot2^4+1\cdot2^3+0\cdot2^2+1\cdot2+0=26$. Check: $16+8+2=26$.

Trap: zero is not in $\mathbb{N}$ under these notes' convention. Write $\mathbb{N}_0$ when zero is included. Also, an integer written as a decimal or fraction can still belong to a smaller set, such as $6/3{=}2{\in}\mathbb{Z}$.

Near miss: $\sqrt{4}$ is written with a root sign, but its value is the natural number 2; $\sqrt{2}$ has no fraction form at all. Decide which set a number belongs to by its value, not by how it is written.

Recall check · answer before opening

1. Why is $\sqrt{2}$ not rational?
2. What integer does $[101101]_2$ represent?

Answers: see "2.1 Number Systems" under Answers at the end.

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2.2 Real Numbers

The real-number rules explain why algebraic transformations are legitimate and why limits do not leave gaps in the number line.

> **In plain words:** the real numbers are the number line with no holes. The field axioms say you can add, subtract, multiply and divide (except by 0) with the usual rules; the order axioms say how $<$ behaves; and **completeness** says every non-empty set that is bounded above has a smallest upper bound, its **supremum**. That last rule is what makes $\sqrt{2}$ a real number.

Field and order

$$\mathbb{R} \text{ is a complete ordered field}$$

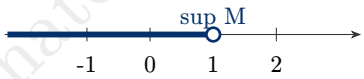
The notes list 9 field axioms, 4 order axioms and 1 completeness axiom. A field has 0, 1, additive inverses and multiplicative inverses for non-zero elements.

The field axioms encode commutativity, associativity, neutral elements, inverses and distributivity. The order axioms make $<$ a total, transitive order compatible with addition and multiplication by positive numbers. Together these support the familiar algebra of inequalities. $\mathbb{Z}$ is not a field because, for example, $1/3$ is not an integer. $\mathbb{Q}$ is an ordered field, but it is not complete.

Upper bound and supremum

$$S = \sup(M) \text{ iff (i) } x \le S \text{ for every } x \in M \text{ and (ii) } S \le T \text{ for every upper bound } T \text{ of } M$$

Read it as: "S is the supremum of M if no element of M is bigger than S, and S is no bigger than any other upper bound": the smallest number that sits on or above the whole set.
The supremum is the least upper bound. It need not belong to M. An infimum is the greatest lower bound.



$\mathbf{M = \{x : x < 1\}}$ runs up to 1 without reaching it. 1 is its supremum; the hollow end shows that 1 is not in M, so M has no maximum.

For M={1, 1/2, 1/3,…}, sup(M)=1 and inf(M)=0. The maximum is 1 because 1∈M; there is no minimum because 0∉M. The completeness axiom says every non-empty real subset bounded above has a supremum in $\mathbb{R}$. It is the property that fills the "gaps" missing in $\mathbb{Q}$.

n-th root from completeness

$$\text{root}_n(x) = \sup\{r \in \mathbb{Q} : r \ge 0 \text{ and } r^n \le x\}, \text{ for } x \ge 0$$

$\text{root}_n(x)$ is the unique non-negative real number whose n-th power is x.

> **Worked example · identify bounds and extrema**
> Let M={1−1/n : n∈ℕ}. Its first values are 0, 1/2, 2/3, 3/4, … . Every value is below 1, so 1 is an upper bound. Values approach 1, so no smaller number is an upper bound: sup(M)=1. But 1∉M, so M has no maximum. The smallest member is 0, hence inf(M)=min(M)=0.

Trap: a supremum is not automatically a maximum. First establish the least or greatest bound, then check whether it belongs to the set.

Near miss: the set $M = \{x : x < 1\}$ has upper bounds 1, 2, 10 and so on, and its supremum is 1, but 1 is not in M, so M has no maximum. A bounded set always has a supremum; it may not have a maximum.

Recall check · answer before opening

1. What two conditions define sup(M)?
2. Why does $\mathbb{Q}$ fail the completeness axiom?

Answers: see "2.2 Real Numbers" under Answers at the end.

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2.3 Absolute Values and Inequalities

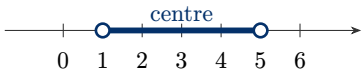
Absolute value turns signs into distance, while the order axioms let you solve distance constraints as intervals.

In plain words: absolute value is distance: $|x|$ is how far x is from 0, and $|x - a|$ is how far x is from a. So an inequality such as $|x - 3| < 2$ simply says "x is less than 2 away from 3", which is the interval $(1, 5)$.

Absolute value

$$|x| = x \text{ if } x \geq 0$$
$$|x| = -x \text{ if } x < 0$$

Read it as: "the absolute value of x is x when x is not negative, and minus x when x is negative": either way the result is not negative.
|x| is distance from x to 0; |x−a| is distance from x to a. In particular, $\sqrt{x^2}$=|x|.



$|x - 3| < 2$ is every x less than 2 away from 3: the open interval $(1, 5)$. The ends are hollow because a distance of exactly 2 is not allowed.

Absolute-value laws

$$|xy| = |x||y|$$
$$||x| - |y|| \leq |x + y| \leq |x| + |y|$$

Read it as: "the absolute value of a product is the product of the absolute values", and the triangle inequality: "|x + y| is at most |x| + |y|".
The right-hand inequality is the triangle inequality. Equality is possible.

For c≥0, |x−a|≤c is equivalent to a−c≤x≤a+c. It describes the closed interval of points no more than c from a. For c>0, |x−a|<c gives an open interval. An inequality with a "greater than" sign usually describes the outside of an interval. Splitting at the point where the expression changes sign is a reliable general method.

Worked example · solve a distance inequality
Solve |2x−1|≤5. By the two-sided form, −5≤2x−1≤5. Add 1: −4≤2x≤6. Divide by 2: −2≤x≤3. The centre is 1/2 and the radius is 5/2, consistent with the interval endpoints.

Sign check: squaring both sides can introduce solutions unless both sides are known to be non-negative. Also, $\sqrt{x^2}$ is |x|, not always x.

Near miss: solving $|x| = -2$ with "$x = 2$ or $x = -2$" looks routine, but a distance cannot be negative, so there is no solution. Check the sign of the right-hand side first.

Recall check · answer before opening

1. What does $|x-a| \leq c$ mean geometrically?
2. What is $\sqrt{x^2}$ for every real x?

Answers: see "2.3 Absolute Values and Inequalities" under Answers at the end.

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2.4 Sums, Products, Binomial Coefficient

Summation and product notation compress repeated operations. Factorials count arrangements; binomial coefficients count selections and become the weights in expansions.

> **In plain words:** $\sum$ is shorthand for "add up" and $\prod$ for "multiply together". The index under the sign says where to start, the number on top where to stop, and the expression after the sign is what to take for each value of the index. $n!$ counts the orders in which n different things can be lined up; the binomial coefficient counts the ways to choose k of them when order does not matter.

Summation and product

$$\sum_{k=m}^{n} a_k = a_m + \ldots + a_n$$

$$\prod_{k=m}^{n} a_k = a_m \cdot \ldots \cdot a_n$$

Read it as: "the sum of a k for k from m to n": add a_m, then a_{m+1}, and so on up to a_n.
Read the index bounds inclusively. A constant factor c can be taken outside a sum; for a product it becomes c raised to the number of factors.

Calculation rules for Σ and Π

$$\text{Split}: \sum_{k=1}^{n} a_k = a_1 + \sum_{k=2}^{n} a_k$$

$$\text{Reverse}: \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_{n+1-k}$$

$$\text{Index shift}: \sum_{k=1}^{n} a_k = \sum_{l=2}^{n+1} a_{l-1}$$

$$\text{Constant}: \sum_{k=m}^{n} a = (n-m+1)a$$

$$\prod_{k=m}^{n} a = a^{n-m+1}$$

$$\text{Telescoping}: \sum_{k=m}^{n} (a_k - a_{k+1}) = a_m - a_{n+1}$$

$$\prod_{k=m}^{n} a_k / a_{k+1} = a_m / a_{n+1}$$

Splitting, reversing, separating $\Sigma(a_k+b_k)$, double sums and index shifts carry over to Π with + replaced by ·. A constant factor does not: $\prod_{k=1}^{5}(2k)=2^5\prod_{k=1}^{5}k$. Check a telescoping sum with m=n: it is the single term a_n-a_{n+1}.

Worked example · a telescoping sum
Evaluate $\sum_{k=1}^{n} 1/(k(k+1))$.

1. Split each term: $1/(k(k+1)) = 1/k - 1/(k+1)$, so with $a_k = 1/k$ it is $a_k - a_{k+1}$.
2. Write out the sum: $(1-1/2)+(1/2-1/3)+\ldots+(1/n-1/(n+1))$. Every inner term cancels.
3. What remains is $a_1 - a_{n+1} = 1-1/(n+1) = n/(n+1)$.
4. Check: $n=1$ gives $1/2$, and $1/(1\cdot 2)=1/2$. §2.4

Factorial and binomial coefficient

$$n! = \prod_{k=1}^{n} k,\ 0! = 1$$

$$C(n,k) = n!/[k!(n-k)!]$$

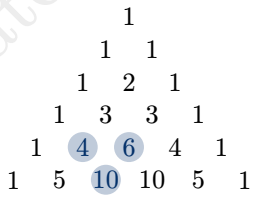
Read it as: "n factorial is the product of k for k from 1 to n", so 4! = 1 · 2 · 3 · 4 = 24.
For $0 \le k \le n$. C(n,k) counts k-element subsets chosen from n distinct elements, with order ignored. n! counts the orderings (permutations) of n distinct elements: {1,2,3} has 3!=6.

Pascal identity and binomial theorem

$$C(n,k) + C(n,k+1) = C(n+1,k+1)$$

$$(a+b)^n = \sum_{k=0}^{n} C(n,k) a^{n-k} b^k$$

The Pascal identity is valid for $0 \le k \le n-1$. The notes state $a,b \neq 0$; the polynomial identity also holds when either is zero. The theorem gives n+1 terms; powers of a descend while powers of b rise.



Pascal's triangle: each entry is the sum of the two above it, as in 4 + 6 = 10, which is C(4, 1) + C(4, 2) = C(5, 2).

Useful coefficient properties are C(n,0)=C(n,n)=1 and C(n,k)=C(n,n−k). Pascal's identity says adjacent entries combine to form the next row. The binomial theorem generalises the familiar square and cube patterns.

Worked example · expand and count
Expand $(x+2)^4$. The row of coefficients is 1, 4, 6, 4, 1, so the terms are x^4, $4x^3\cdot 2$, $6x^2\cdot 2^2$, $4x\cdot 2^3$, and 2^4. Thus $(x+2)^4 = x^4+8x^3+24x^2+32x+16$. Check the endpoints: the leading term is x^4, and the constant is 16.
Separately, the number of 3-person groups from 8 people is $C(8,3) = 8\cdot 7\cdot 6/(3\cdot 2\cdot 1) = 56$; ordering the group members would overcount each group 3! times.

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Memory hook: in the term indexed by k, the exponent on a is n−k and the exponent on b is k. Check that each term's exponents add to n.

Near miss: "the sum of k^2 for $k = 1$ to 3" looks as if it might be $(1 + 2 + 3)^2 = 36$, but it means $1^2 + 2^2 + 3^2 = 14$. The expression after $\sum$ is worked out for each k first, then the results are added.

Recall check · answer before opening

1. How does C(n,k) differ from a permutation count?
2. What are the coefficients in the expansion of $(a+b)^4$?

Answers: see "2.4 Sums, Products, Binomial Coefficient" under Answers at the end.

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2.5 Mathematical Induction

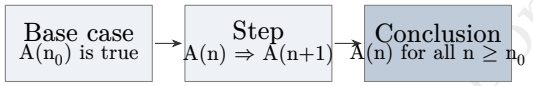
Induction proves infinitely many linked claims using a finite argument: establish a starting case, then show each true case passes the claim to the next one.

> **In plain words:** induction works like a row of dominoes. Show that the first one falls (the **base case**), and show that whenever one falls it knocks over the next (the **step**). Then every domino falls, so the claim holds for every n from the starting value on.

Induction principle

$$\text{Prove } A(n_0), \text{ and prove } A(n) \Rightarrow A(n+1) \text{ for arbitrary } n \geq n_0$$

Read it as: "if $A(n_0)$ is true, and $A(n)$ being true always makes $A(n+1)$ true, then $A(n)$ is true for every $n \geq n_0$".
Then A(n) holds for every integer $n \geq n_0$. State the starting index and keep the induction hypothesis available in the step.



Induction as a chain: the base case starts it and the step passes it on, so it reaches every n from n_0 on.

Write the induction step as a conditional: assume A(n) for an arbitrary allowed n, and use it to prove A(n+1). Do not assume the result you are trying to prove. The base case starts the chain; the implication advances it. For two-step induction, verify two consecutive base cases and advance from n to n+2, which covers both parities. In **strong induction** (the notes' "extended induction hypothesis") you assume $A(n_0)$, $A(n_0+1)$, …, A(n) all hold and use any of them to prove A(n+1); this helps when the next case depends on an earlier case rather than only on the one before.

Sums proved by induction

$$\sum_{k=1}^{n} k = n(n+1)/2$$

$$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$$

$$\sum_{k=1}^{n} k^3 = (n(n+1)/2)^2$$

Each is proved with base case n=1 and the step from n to n+1. Check n=2: 3, 5 and 9.

Bernoulli's inequality

$$(1+x)^n \geq 1 + \text{nx}, \text{ for } n \in \mathbb{N} \text{ and } x \geq -1$$

In the notes' first proof, x is restricted to $x \geq -1$; this makes 1+x non-negative so multiplying preserves the inequality.

Extended Bernoulli inequality

$$(1+x)^n \geq 1+\mathrm{nx}, \text{ for } n \in \mathbb{N} \text{ and } x \geq -2$$

Use two-step induction: prove n=1 and n=2, then advance from n to n+2.

In the extended step, $(1+x)^{n+2}=(1+x)^2(1+x)^n$. The square factor is non-negative. Applying the hypothesis and expanding leaves the extra term $x^2(2n+1+nx)$, which is non-negative for $x\geq -2$ and $n\geq 1$.

Worked example · sum the first n integers

Claim: $1+2+\ldots+n=n(n+1)/2$ for $n\in\mathbb{N}$.

1. **Base:** n=1 gives $1=1\cdot 2/2$.
2. **Hypothesis:** assume $1+\ldots+n=n(n+1)/2$.
3. **Step:** $1+\ldots+n+(n+1)=n(n+1)/2+(n+1)=(n+1)(n+2)/2$.
4. This is the claimed formula with n replaced by n+1, so it holds for all $n\in\mathbb{N}$.

Trap: checking several values is not an induction proof. In the step, n is arbitrary—not one particular example—and the induction hypothesis must be stated before it is used.

Near miss: $n^2 + n + 41$ is prime for $n = 0, 1, 2, \ldots, 39$, yet at $n = 40$ it gives $1681 = 41^2$. Forty checked cases are still not a proof; only the step from n to $n + 1$ covers every case.

Recall check · answer before opening

1. What is the logical role of the induction hypothesis?
2. Why does proving $A(n)\Rightarrow A(n+1)$ for arbitrary n not suffice without a base case?

Answers: see "2.5 Mathematical Induction" under Answers at the end.

Formula map and checks

Topic	Relation	Check
Number systems	$\mathbb{N}\subsetneq\mathbb{Z}\subsetneq\mathbb{Q}\subsetneq\mathbb{R}$	Does the value have a ratio of integers?
Supremum	least upper bound of M	Upper bound and least among all upper bounds.
Absolute value	$\lvert x\rvert\geq 0$; $\lvert xy\rvert=\lvert x\rvert\lvert y\rvert$	Distance cannot be negative; check signs.
Triangle inequality	$\lvert x+y\rvert\leq\lvert x\rvert+\lvert y\rvert$	Try equality and opposite signs.
Counting	$C(n,k)=n!/[k!(n-k)!]$	Order ignored; $0\leq k\leq n$.
Binomial theorem	$(a+b)^n=\sum_{k=0}^{n} C(n,k)a^{n-k}b^k$	n+1 terms; powers in each term add to n.
Induction	$A(n_0)$; $A(n)\Rightarrow A(n+1)$	Base case and arbitrary induction step.

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Derive it cold

Rebuild these on blank paper before opening the answers in the relevant sections.

1. Prove $\sqrt{2}$ is irrational by contradiction. §2.1
2. Derive $\sqrt{x^2}=|x|$ by splitting $x\geq 0$ and $x<0$. §2.3
3. Derive the factorial expression for C(n,k) by counting ordered selections and dividing by k!. §2.4
4. Prove the sum $1+\ldots+n=n(n+1)/2$ by induction. §2.5
5. Use the induction hypothesis and Pascal's identity to prove the binomial theorem for n+1 from n. §2.4–2.5

Explain it

Answer aloud in complete sentences. Then compare with the model answer. (Retrieval and explanatory questioning [MIS, AMN, NEW].)

1. **Why** does being expressible as p/q not guarantee a unique written fraction?
2. **Why** are field and order axioms not enough to define $\mathbb{R}$?
3. **When** is a supremum a maximum?
4. **Why** is $\sqrt{x^2}$ not always x?
5. **When** should you use a binomial coefficient instead of a permutation count?
6. **Why** does induction need both a base case and a step?

Answers: see "Explain it" under Answers at the end.

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Mixed practice

Try all 25 problems before opening the answer key. They are interleaved, so identify the method before calculating. (Desirable difficulty and varied retrieval [MIS, AMN].)

1. Classify $-5/2$ as natural, integer, rational, irrational or real; list every applicable category.
2. Solve $|x-4|<3$ and write the answer as an interval.
3. Prove by induction that $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ for $n \in \mathbb{N}$.
4. Expand $(a+b)^3$.
5. For $M=\{1-1/n : n \in \mathbb{N}\}$, find $\sup(M)$, $\max(M)$, $\inf(M)$, and $\min(M)$.
6. Evaluate $\sum_{k=1}^{4}(2k-1)$ and $\prod_{k=1}^{4} k$.
7. In one sentence, state the contradiction in the proof that $\sqrt{2}$ is irrational.
8. Solve $|3x-2|>7$.
9. Why is $\mathbb{Z}$ not a field under ordinary addition and multiplication? Give one specific failed field property.
10. How many 2-element subsets can be chosen from an 8-element set?
11. Convert $[101101]_2$ to base ten.
12. Check the triangle inequality for $x=-7$, $y=3$. Compute both sides.
13. Prove by induction that 7 divides 8^n-1 for $n \in \mathbb{N}$.
14. For $M=\{1/n : n \in \mathbb{N}\}$, give its supremum and infimum. Say which are attained.
15. Compute $C(7,3)$, showing the cancellation.
16. Convert $[10.11]_2$ to base ten.
17. Solve $|2x+1| \geq 5$ by cases and check endpoints.
18. In an ordered field, if $a<b$, what can you conclude about $a+c$ and $b+c$? What if you multiply by $c<0$?
19. Use Pascal's identity to evaluate $C(6,2)+C(6,3)$.
20. Use Bernoulli's inequality to give a lower bound for $(1.1)^5$.
21. What is the principal real fourth root of 16?
22. Evaluate $\sum_{k=1}^{n} 1/(k(k+1))$ by writing each term as $1/k-1/(k+1)$.
23. Evaluate $\prod_{k=2}^{4} k$.
24. Prove by induction that $1+3+\ldots+(2n-1)=n^2$.
25. Find the coefficient of x^2y^3 in $(x+y)^5$.

Answers: see "Mixed practice" under Answers at the end.

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Exam-style problems

Allow about 20 minutes for each. Work every part before checking the answer. Afterward, note where you lost a step and revisit the cited sections.

1 · Bounds, distance and counting

Let $M=\{x\in\mathbb{R}: |x-2|\le 3\}$ and let S be the set of 2-element subsets of $\{1,2,3,4,5\}$.

1. Write M as an interval and give its minimum, maximum, infimum and supremum.
2. Find all $x\in M$ for which $|x|>2$.
3. How many members does S have? Explain why order does not matter.
4. Find the coefficient of x^3y^2 in $(x+y)^5$.

Answers: see "Exam-style problems" under Answers at the end.

2 · Prove the binomial theorem

Let P(n) be the claim $(a+b)^n=\sum_{k=0}^{n} C(n,k)a^{n-k}b^k$, for non-zero real a,b and $n\in\mathbb{N}$.

1. Verify P(1).
2. Assume P(n). Multiply its right-hand side by (a+b) to obtain the expression for $(a+b)^{n+1}$.
3. Group adjacent terms and use Pascal's identity to prove P(n+1).
4. Use the theorem to expand $(x+2)^4$.

Answers: see "Exam-style problems" under Answers at the end.

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Slips in the notes

These are source-checked issues and the corresponding precise statement used in this guide. Printed page numbers refer to the lecture notes.

1. **Printed page 12, missing condition:** the remark says that for arbitrary rational a,b, the equation $ax=b$ always has a solution $x\in\mathbb{Q}$. This fails when $a=0$ and $b\neq 0$. The statement is correct when $a\neq 0$; for $a=0$ it has no solution if $b\neq 0$ and every rational x is a solution if $b=0$.
2. **Printed page 18, error:** the telescoping rule is printed as $\sum_{k=m}^{n}(a_k-a_{k+1})=a_m-a_n$. Writing out the terms leaves a_m-a_{n+1}, and the product rule likewise gives a_m/a_{n+1}. For $m=n$ the printed form gives 0 instead of a_n-a_{n+1}. This guide uses the corrected rules.
3. **Printed page 19, missing condition:** Theorem 2.13(iii) is stated for $k=0,\ldots,n$, but at $k=n$ it needs $C(n,n+1)$, which the notes do not define. This guide states Pascal's identity for $0\le k\le n-1$ (or set $C(n,k)=0$ for $k>n$).
4. **Printed page 15, loose wording:** "for $\mathbb{R}^2$ and $\mathbb{C}$ the order axioms do not hold" means that no ordering of $\mathbb{C}$ satisfies all of (A10)–(A13) together with the field operations. $\mathbb{C}$ can be totally ordered, for example lexicographically, but such an order is not compatible with multiplication (A13).

Self-check

Tick a skill only in a later session, at least one day after learning it, when you can show it without notes. Use the problem numbers as evidence.

- ☐ Classify values in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ and distinguish irrational numbers. (1, 7)
- ☐ Convert binary integers and fixed-point numbers. (11, 16)
- ☐ Rebuild the contradiction proof for $\sqrt{2}$. (7)
- ☐ Explain why $\mathbb{Q}$ is an ordered field but not complete. (9, 18, 21)
- ☐ Find bounds, suprema, infima, maxima and minima. (5, 14)
- ☐ Solve absolute-value inequalities with correct endpoints. (2, 8, 17)
- ☐ Apply the triangle inequality and check it numerically. (12)
- ☐ Evaluate and interpret summation and product notation, including telescoping sums. (6, 22, 23)
- ☐ Use factorials and binomial coefficients for counting. (10, 15)
- ☐ Use Pascal's identity and expand a binomial. (4, 19, 25)
- ☐ Write complete induction proofs from the correct base index. (3, 13, 24)
- ☐ Apply Bernoulli's inequality with its stated condition. (20)
- ☐ Rebuild the listed derivations without notes.
- ☐ Solve both multi-part problems and justify every step.

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Answers

Start from memory

1. $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$; the irrational reals are $\mathbb{R}\backslash\mathbb{Q}$. §2.1
2. Completeness: each non-empty real subset that is bounded above has a least upper bound in $\mathbb{R}$. §2.2
3. The distance between x and a. §2.3
4. The binomial coefficient (n choose k)=n!/(k!(n−k)!). §2.4
5. Prove a base case, then prove A(n) implies A(n+1) for arbitrary n in the stated domain. §2.5

2.1 Number Systems

1. Assuming $\sqrt{2}$=p/q in lowest terms forces both p and q even, a contradiction. §2.1
2. 32+8+4+1=45. §2.1

2.2 Real Numbers

1. It is an upper bound, and it is no larger than any other upper bound. §2.2
2. The rationals bounded by $\sqrt{2}$ have no supremum in $\mathbb{Q}$ because their least upper bound is the irrational $\sqrt{2}$. §2.2

2.3 Absolute Values and Inequalities

1. x lies within distance c of a; for c≥0, a−c≤x≤a+c. §2.3
2. |x|. §2.3

2.4 Sums, Products, Binomial Coefficient

1. It counts selections where order does not matter; permutations count orderings. §2.4
2. 1, 4, 6, 4, 1. §2.4

2.5 Mathematical Induction

1. It is the assumption used to derive the next case; it is not itself the conclusion. §2.5
2. Without a true starting case, the implication could advance a chain that never begins. §2.5

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Explain it

1. Many numerator-denominator pairs represent the same quotient, such as 2/3 and 4/6. Requiring coprime numerator and denominator gives the reduced representation used in the notes.
2. $\mathbb{Q}$ satisfies the field and order rules too, but it has gaps. Completeness distinguishes $\mathbb{R}$ by guaranteeing least upper bounds for non-empty bounded-above subsets.
3. When the supremum belongs to the set. A supremum can bound a set without being one of its elements, as with $\{1-1/n\}$.
4. The square root is non-negative. If x is negative, the non-negative number whose square is x^2 is $-x$, so the result is $|x|$.
5. Use C(n,k) when selecting k distinct objects from n and order does not matter. If different orders count as different outcomes, use a permutation count.
6. The base case starts the chain. The step says every true case passes truth to the next one; together they cover every index from the starting point onward.

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Mixed practice

1. It is rational and real; it is not natural or an integer. §2.1
2. Distance from 4 is less than 3, so $1<x<7$; solution $(1,7)$. §2.3
3. Base n=1: $1=1\cdot2\cdot3/6$. Assume the sum through n is $n(n+1)(2n+1)/6$ and add $(n+1)^2$. Factoring out $(n+1)/6$ leaves $n(2n+1)+6(n+1)=2n^2+7n+6=(n+2)(2n+3)$, so the sum is $(n+1)(n+2)(2n+3)/6$, the formula at n+1. §2.5
4. $a^3+3a^2b+3ab^2+b^3$. §2.4
5. sup=1, no maximum; inf=min=0. The values approach 1 but never equal it, while n=1 gives 0. §2.2
6. $1+3+5+7=16$; $1\cdot2\cdot3\cdot4=24$. §2.4
7. Writing $\sqrt{2}=p/q$ in lowest terms forces p even from $p^2=2q^2$, then q even after substituting $p=2m$. That contradicts coprimality. §2.1
8. $3x-2>7$ or $3x-2<-7$, so $x>3$ or $x<-5/3$. §2.3
9. Non-zero integers need not have multiplicative inverses in $\mathbb{Z}$: 2 has inverse $1/2$, which is not an integer. §2.2
10. $C(8,2)=8\cdot7/(2\cdot1)=28$. §2.4
11. $32+8+4+1=45$. §2.1
12. $|x+y|=|-4|=4$ and $|x|+|y|=7+3=10$; $4\le10$. §2.3
13. Base n=1: $8-1=7$. Assume $8^n-1=7m$. Then $8^{n+1}-1=8(8^n-1)+7=56m+7$, divisible by 7. §2.5
14. sup=max=1; inf=0, not attained. Every term is positive and tends to 0. §2.2
15. $C(7,3)=7\cdot6\cdot5/(3\cdot2\cdot1)=35$. §2.4
16. $2+1/2+1/4=2.75$. §2.1
17. $2x+1\ge5$ gives $x\ge2$; $2x+1\le-5$ gives $x\le-3$. Include equality in both cases: $(-\infty,-3]\cup[2,\infty)$. §2.3
18. $a+c<b+c$. Multiplying by a negative reverses direction: $ac>bc$. §2.2
19. By Pascal, $C(6,2)+C(6,3)=C(7,3)=35$. §2.4
20. Bernoulli with $x=0.1, n=5$ gives $(1.1)^5\ge1+5(0.1)=1.5$. §2.5
21. The principal fourth root is 2, the unique non-negative real whose fourth power is 16. (-2) is another fourth root, but not the principal root. §2.2
22. It telescopes with $a_k=1/k$: $a_1-a_{n+1}=1-1/(n+1)=n/(n+1)$. §2.4
23. $2\cdot3\cdot4=24$. §2.4
24. Base n=1: $1=1^2$. Assume the sum through n is n^2; adding $2n+1$ gives $n^2+2n+1=(n+1)^2$. §2.5
25. The term is indexed by k=3, so its coefficient is $C(5,3)=10$. §2.4

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Exam-style problems

1. $|x-2|\le 3$ means $-3\le x-2\le 3$, hence $-1\le x\le 5$. Thus $M=[-1,5]$, and $\min=\inf=-1$, $\max=\sup=5$. §2.2 §2.3
2. Within $[-1,5]$, the condition $|x|>2$ leaves $x>2$, so $(2,5]$. §2.3
3. $|S|=C(5,2)=5\cdot 4/(2\cdot 1)=10$. Each pair of distinct elements describes one subset regardless of order. §2.4
4. The $k=2$ term is $C(5,2)x^3y^2$, so the coefficient is 10. §2.4

Exam-style problems

1. For $n=1$, $C(1,0)a+C(1,1)b=a+b=(a+b)^1$. §2.4 §2.5
2. By $P(n)$, $(a+b)^{n+1}=\sum_{k=0}^{n} C(n,k)a^{n-k}b^k(a+b)$. Distributing gives terms with one extra a and terms with one extra b. §2.4 §2.5
3. For each interior power, the coefficient is $C(n,k)+C(n,k-1)=C(n+1,k)$, by Pascal's identity with the index shifted. The endpoint coefficients are 1. Thus the sum is $\sum_{k=0}^{n+1}C(n+1,k)a^{n+1-k}b^k$; $P(n+1)$ follows. The base and implication prove the theorem for all $n\in\mathbb{N}$. §2.4 §2.5
4. The coefficients 1,4,6,4,1 give $x^4+8x^3+24x^2+32x+16$. §2.4

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